LAUMON SPACES AND THE
CALOGERO-SUTHERLAND INTEGRABLE SYSTEM

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1. INTRODUCTION

This paper is concerned with the Laumon quasiflag spaces $\mathcal{M}_\gamma$, which parametrize flags of the form

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}_{n-1} \subset \mathcal{O}_{\mathbb{P}^1}^n \tag{1.1}$$

In the above, each $\mathcal{F}_i$ is a torsion-free sheaf of rank i on $\mathbb{P}^1$ whose fiber at $\infty \in \mathbb{P}^1$ is fixed beforehand. The degrees of the sheaves $\mathcal{F}_i$ are predetermined by the index γ, which lies in a monoid Q^- in the root lattice of $\mathfrak{sl}_n$ (see Section 4 for the exact definitions). The maximal torus $T \subset SL_n$ acts on each $\mathcal{M}_\gamma$ by linear transformations of the ambient sheaf $\mathcal{O}_{\mathbb{P}^1}^n$, while $\mathbb{C}^*$ acts on $\mathcal{M}_\gamma$ by multiplying the base $\mathbb{P}^1$ of the sheaves. The resulting $T \times \mathbb{C}^*$ action on $\mathcal{M}_\gamma$ will give us equivariant cohomology classes.

We will study the generating function

$$Z(m) = \sum_{\gamma \in Q^-} e^\gamma \int_{\mathcal{M}_\gamma} c(\mathcal{T}\mathcal{M}_\gamma, mx) \tag{1.2}$$

In the above, $c(\mathcal{T}\mathcal{M}_\gamma, \cdot)$ denotes the $T \times \mathbb{C}^*-$equivariant Chern polynomial of the tangent bundle of $\mathcal{M}_\gamma$, $m \in \mathbb{C}$ is a parameter and x is the standard coordinate on $\mathbb{C}$. This generating function was first introduced by Nekrasov in [20], in the more complicated setting when $\mathfrak{sl}_n$ is replaced by its affine counterpart $\widehat{\mathfrak{sl}_n}$.

In our $\mathfrak{sl}_n$ situation, Braverman ([5]) has conjectured that $Z(m)$ is very closely related to the *quantum trigonometric Calogero-Sutherland integrable system*. In fact, in [4] he proves the $m \to \infty$ limit case of this conjecture. Explicitly, Braverman considers the generating function

$$Z = \sum_{\gamma \in Q^-} e^\gamma \int_{\mathcal{M}_\gamma} 1 \tag{1.3}$$

and shows that Z is the eigenfunction of the *quantum Toda hamiltonian* (up to a constant factor). We will prove Braverman's conjecture for general m, namely that $Z(m)$ equals the eigenfunction of the quantum trigonometric Calogero-Sutherland hamiltonian, up to a factor which we will specify.

The function $Z(m)$ is expected to be closely related to the equivariant $J-$function of the quantum cohomology of the cotangent bundle of the complete flag variety of $\mathbb{C}^n$. The affine analogue of $Z(m)$ appears in $\mathcal{N} = 2$ super-symmetric $4-$dimensional gauge theory with adjoint matter, and is conjecturally very closely

related to the *quantum elliptic Calogero-Moser integrable system*. Both of these directions are described in [5].

Andrei Okounkov has suggested that one can study $Z(m)$ by modeling it as the character of a certain geometric operator $A(m)$. In this paper, we follow this idea by relating $A(m)$ to $\mathfrak{sl}_n-$intertwiners. Then we use the theory of generalized characters developed by Etingof, Frenkel and Kirillov to compute $Z(m)$. In this way we obtain:

$$Z(m) = Y_{\frac{a}{x},m} \cdot e^{-\frac{a}{x}} \cdot \left(\prod_{\alpha \in R^+} \frac{1}{1 - e^{-\alpha}} \right)^{m+1} \tag{1.4}$$

where a, x are standard coordinates on T and $\mathbb{C}^*$ (respectively). In the above, $Y_{\frac{a}{x},m}$ is the eigenfunction of the Calogero-Sutherland hamiltonian with eigenvalue $\frac{(a,a)}{x^2}$ and highest term $e^{\frac{a}{x}}$.

Let us say a few words about the structure of the paper: in Section 2 we introduce certain notations and concepts pertaining to Lie algebras, Verma modules, differential operators and quiver representations. In Section 3 we give a short presentation of equivariant cohomology with respect to the action of a compact Lie group. In Section 4 we consider the moduli spaces $\mathcal{M}_\gamma$ in detail and establish some of their properties. In Section 5, we introduce a very important vector bundle E and give several interpretations of it. In Section 6 we use the bundle E to construct the operator $A(m)$ and relate its character to $Z(m)$. In Section 7 we use the theory of generalized characters of intertwiners (see, for example, [14]) to compute the character of $A(m)$. We prove relation (1.4) in Theorem 7.5. Finally, as a corollary, we take the limit as $m \to \infty$ and obtain the finite-dimensional statement of Corollary 3.7 of [4].

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2. Basic Definitions

2.1. In this section we will describe some aspects of the representation theory of the simple Lie algebra $\mathfrak{sl}_n$. Fix a basis $w_1, ..., w_n$ of $\mathbb{C}^n$. To this choice of basis, there corresponds a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_n$ and a root system $R \subset \mathfrak{h}^*$. Corresponding to R we have the standard decomposition

$$\mathfrak{sl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

Explicitly, $\mathfrak{sl}_n$ consists of traceless $n \times n$ complex matrices, $\mathfrak{h}$ consists of traceless diagonal matrices and $\mathfrak{n}^-/\mathfrak{n}^+$ consists of strictly lower/upper triangular $n \times n$

matrices (with respect to the basis $w_1, ..., w_n$).

The root system R consists of vectors $w_i - w_j$ $(1 \le i \ne j \le n)$. If $i < j$ the corresponding root is called positive, while if $i > j$ it is called negative. The set of positive roots is denoted by R^+, while the set of negative roots is denoted by R^-. The half-sum of the positive roots will be denoted by

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$$

Further, $\alpha_i = w_i - w_{i+1} \in R^+$ will be called the simple roots. Inside $\mathfrak{h}^*$ we have the root lattice

$$Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$$

Letting $\mathbb{Z}_- = \{0, -1, -2, ...\}$, the lattice Q contains the monoid

$$Q^- = \bigoplus_{i=1}^{n-1} \mathbb{Z}_-\alpha_i$$

One can impose a partial order on weights $\lambda \in \mathfrak{h}^*$ by setting $\lambda \le \mu$ if $\lambda - \mu \in Q^-$.

The vector spaces $\mathfrak{n}^-, \mathfrak{h}, \mathfrak{n}^+$ have standard bases

$$\mathfrak{n}^- = \langle f_\alpha, \alpha \in R^+ \rangle$$

$$\mathfrak{h} = \langle h_1, ..., h_{n-1} \rangle$$

$$\mathfrak{n}^+ = \langle e_\alpha, \alpha \in R^+ \rangle$$

Explicitly, if $\alpha = w_i - w_j$ with $i < j$, then f_α is the matrix with 1 at the intersection of row j and column i and 0 everywhere else, while e_α is the matrix with 1 at the intersection of row i and column j and 0 everywhere else. When $\alpha = w_i - w_{i+1}$ is a simple root, we write $f_i = f_{\alpha_i}$ and $e_i = e_{\alpha_i}$. Note that the f_i's (respectively, the e_i's) generate $\mathfrak{n}^-$ (respectively, $\mathfrak{n}^+$) as Lie algebras. Finally, one defines

$$h_i = [e_i, f_i].$$

2.2. Given $\beta \in \mathfrak{h}^*$ and a set of linearly independent vectors $\beta_1, ..., \beta_t \in \mathfrak{h}^*$, we define a *cone* to be the set

$$C = \{\lambda | \lambda = \beta + k_1\beta_1 + ... + k_t\beta_t, \ k_1, ..., k_t \in \mathbb{N}_0\} \subset \mathfrak{h}^* \tag{2.1}$$

An example of a cone is Q^- defined in the previous subsection. An $\mathfrak{sl}_n$−module V is said to posses a *conic weight decomposition* if for some cone $C \subset \mathfrak{h}^*$, one has

$$V = \bigoplus_{\lambda \in C} V[\lambda] \tag{2.2}$$

where each $V[\lambda]$ is a finite-dimensional subspace such that Cartan elements $h \in \mathfrak{h}$ act on it as multiplication by $\langle h, \lambda \rangle$. The most important example of such a module is the Verma module.

Definition 2.3. We define the Verma module $M(\lambda_0)$ with highest weight λ_0 to be the $\mathfrak{sl}_n-$module freely generated over $U\mathfrak{n}^-$ by a single vector v under the relations

$$e \cdot v = 0, \ \forall e \in \mathfrak{n}^+$$

$$h \cdot v = \langle h, \lambda_0 \rangle v, \ \forall h \in \mathfrak{h}$$

The representation $M(\lambda_0)$ is irreducible for generic λ_0, and it has a conic weight decomposition with highest weight λ_0:

$$M(\lambda_0) = \bigoplus_{\lambda \in \lambda_0 + Q^-} M(\lambda_0)[\lambda].$$

2.3. Consider two $\mathfrak{sl}_n-$modules U and V, where V has a conic weight decomposition as in (2.2). Given a linear operator $\Phi : V \to V \otimes U$, we define its character as the formal $U-$valued expression

$$\chi_\Phi = \sum_{\lambda \in C} e^\lambda \cdot \text{Tr}(\Phi|_{V[\lambda]}) \tag{2.4}$$

Here the e^λ should be perceived as formal commuting symbols (with the standard relation $e^{\lambda_1 + \lambda_2} = e^{\lambda_1} e^{\lambda_2}$). The operator $\Phi|_{V[\lambda]}$ is defined as the restriction of Φ to the finite-dimensional factor $V[\lambda]$, followed by projection onto $V[\lambda] \otimes U$.

As noted in [14], the character can also be perceived as the equivariant function on the Lie group SL_n given by

$$\chi_\Phi(g) = \text{Tr}(\Phi \cdot g)$$

Here, equivariant means that for $x, g \in SL_n$ we have

$$\chi_\Phi(xgx^{-1}) = x \cdot \chi_\Phi(g) \tag{2.5}$$

The equivariance implies that χ is determined by its values on elements e^h of the maximal torus. The two descriptions of χ_Φ presented above are related by the fact that

$$\chi_\Phi(e^h) = \sum_{\lambda \in C} e^{\langle h, \lambda \rangle} \cdot \text{Tr}(\Phi|_{V[\lambda]})$$

The above sum converges for h in a certain cone in $\mathfrak{h}$.

2.4. The character defined in (2.4) is a particular example of a *power series*. We define a power series to be an expression of the form

$$\chi = \sum_{\lambda \in C} e^\lambda c_\lambda \tag{2.6}$$

where $C \subset \mathfrak{h}^*$ is a cone, and c_λ are coefficients belonging to some representation U of $\mathfrak{sl}_n$. A particularly important power series is the *Weyl denominator*

$$\delta = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha}) \tag{2.7}$$

and its inverse

$$\delta^{-1} = e^{-\rho} \prod_{\alpha \in R^+} \frac{1}{1 - e^{-\alpha}}$$

Both δ and δ^{-1} are power series with $\pm\rho + Q^-$ as the associated cones. In general, two types of objects can act on power series χ: elements of $\mathfrak{sl}_n$ can act on the coefficients c_λ, and *differential operators* can act on the symbols e^λ. Examples of such differential operators are the partial derivative in the direction of $\alpha \in \mathfrak{h}^*$:

$$\partial_\alpha(e^\lambda) = (\alpha, \lambda)e^\lambda \tag{2.8}$$

and the Laplace operator:

$$\Delta_\mathfrak{h}(e^\lambda) = (\lambda, \lambda)e^\lambda. \tag{2.9}$$

The main differential operator we will be concerned with is the *quantum trigonometric Calogero-Sutherland hamiltonian* ([14]):

$$L(m) = \Delta_\mathfrak{h} - 2m(m+1) \sum_{\alpha \in R^+} \frac{1}{(e^{\alpha/2} - e^{-\alpha/2})^2}$$

Consider a power series

$$Y_{\lambda_0, m} = \sum_{\lambda \in \lambda_0 + Q^-} e^\lambda c_\lambda \tag{2.10}$$

normalized such that $c_{\lambda_0} = 1$. The fact that $Y_{\lambda_0, m}$ is an eigenfunction of $L(m)$ with eigenvalue $s \in \mathbb{C}^*$ is equivalent to the following relations on the coefficients c_λ:

$$c_\lambda \cdot ((\lambda, \lambda) - s) = 2m(m+1) \sum_{\alpha \in R^+} \sum_{j \geq 1} j \cdot c_{\lambda + j\alpha} \tag{2.11}$$

The above expression for $\lambda = \lambda_0$ implies $s = (\lambda_0, \lambda_0)$. Furthermore, for generic $\lambda_0 \in \mathfrak{h}^*$, the above relation recursively determines all coefficients c_λ from $c_{\lambda_0} = 1$. Thus there is a unique eigenfunction of $L(m)$ with highest term e^{λ_0}, and we will henceforth denote it by $Y_{\lambda_0, m}$.

2.5. We need to touch on one more topic from representation theory, which is quite independent of what was discussed above. A *quiver* Q with vertex set I and edge set E is a directed graph. A representation V of Q will be a collection of vector spaces V_i for every $i \in I$, together with linear maps $\phi_e : V_i \to V_j$ for every edge $e \in E$ between vertices i and j. The dimension of a representation V will be the vector of positive integers $\dim(V) = (d_i, i \in I)$, where d_i is the dimension of the vector space V_i. A map of two representations V and V' is a collection of maps between the vector spaces V_i and V_i' (as i ranges over I) that commute with the edge maps.

A simple representation is one that has no non-trivial subrepresentations. An indecomposable representation is one which does not decompose non-trivially into a direct sum. Let S, R and $\text{Rep}(Q)$ denote the sets of isomorphism classes of simple, indecomposable and all representations of the quiver Q, respectively. Any representation is a direct sum of indecomposables in R, and thus $\text{Rep}(Q) = \mathbb{N}[R]$.

Define the *Ringel-Hall algebra* $U_q(Q)$ as the algebra generated by symbols e_κ for $\kappa \in \text{Rep}(Q)$, with multiplication given by the rule

$$e_{\kappa'} \cdot e_{\kappa''} = q^{\langle \dim(\kappa'), \dim(\kappa'') \rangle} \sum_{\kappa \in \text{Rep}(Q)} P^\kappa_{\kappa', \kappa''}(q) e_\kappa$$

where $\langle \cdot, \cdot \rangle$ is the Euler form of the quiver. The definition of $P^\kappa_{\kappa', \kappa''}$ is the following: suppose the ground field is $\mathbb{F}_q$, the finite field with q elements. Then take a representation V from the class κ, and define $P^\kappa_{\kappa', \kappa''}(q)$ to be the number of subrepresentations $V' \subset V$ such that V' is in the isomorphism class κ' and V/V' is in the isomorphism class κ''. The function $P^\kappa_{\kappa', \kappa''}(q)$ is a polynomial in q (see [22]), so the definition of the multiplication law makes sense for q an indeterminate. In particular, we can specialize the multiplication law at $q = 1$ and obtain the algebra $U(Q) := U_1(Q)$.

In this paper, we will only be concerned with the quiver

$$A_{n-1} : 1 \to 2 \to ... \to n - 1$$

with vertex set $I = \{1, 2, ..., n-1\}$. Its simple representations are denoted by $[i; 1)$, where $i \in I$. By definition, $[i; 1)$ is the representation with a one-dimensional vector space at the vertex i and all edge maps 0. The indecomposable representations are denoted by $[i; l)$, where $i \in I$ and $1 \le l \le n - i$. By definition, $[i; l)$ is the representation with a one-dimensional vector space at the vertices $i, i+1, ..., i+l-1$ and maps

$$... \to 0 \to V_i \xrightarrow{\cong} V_{i+1} \xrightarrow{\cong} ... \xrightarrow{\cong} V_{i+l-1} \to 0 \to ...$$

One can explicitly compute the Ringel-Hall algebra $U(A_{n-1})$. If we denote $e_i := e_{[i;1)}$, we have

$$e_{[i;l)} = [e_{i+l-1}, [e_{i+l-2}, [....[e_{i+1}, e_i]...]]]$$

where $[\cdot, \cdot]$ is the standard commutator. For a general isomorphism class $\kappa \in \text{Rep}(A_{n-1})$, we have

$$\kappa = \bigoplus_{i=1}^{n-1} \bigoplus_{l=1}^{n-l} [i; l)^{\oplus k_{il}} \Rightarrow e_\kappa = \prod_{i=1}^{n-1} \prod_{l=1}^{n-l} \frac{e_{[i;l)}^{k_{il}}}{k_{il}!} \tag{2.12}$$

In particular, the above shows us that $U(A_{n-1})$ is generated by $e_1, ..., e_{n-1}$. Let $E_{i,j}$ denote the matrix with 1 at the intersection of row i and column j, and 0 everywhere else. Then we have the following ([22]):

Theorem 2.13. *The assignment $e_i \to E_{i+1,i}$ gives us an isomorphism*

$$U(A_{n-1}) \xrightarrow{\cong} U(\mathfrak{n}^-)$$

where $U(\mathfrak{n}^-)$ is the universal enveloping algebra of $\mathfrak{n}^- \subset \mathfrak{sl}_n$.

3. Equivariant Cohomology

3.1. Our presentation of equivariant cohomology will follow [1]. Let us consider a Lie group G, let BG be the classifying space of G and $\pi : EG \to BG$ the universal $G-$bundle. Given a smooth variety X with a $G-$action, define the space

$$EG \times_G X = EG \times X / \{(pg, x) \sim (p, gx), p \in EG, x \in X, g \in G\}$$

The $G-$equivariant cohomology of X is defined as

$$H^*_G(X) := H^*(EG \times_G X)$$

When the group G is clear from context, we will simply call this construction *equivariant cohomology*.

3.2. Equivariant cohomology satisfies the following properties:

- *existence of pull-backs:*   if $f : X \to Y$ is a $G-$equivariant map, then there exists a pull-back homomorphism $f^* : H^*_G(Y) \to H^*_G(X)$.
- *module structure:*   if we consider $\pi : X \to pt$, then $\pi^* : H^*_G(pt) \to H^*_G(X)$ endows $H^*_G(X)$ with a structure of a $H^*_G(pt)-$module.
- *cohomology of the point:* $H^*_G(pt) = \mathbb{C}[x_1, ..., x_n]$, where $x_1, ..., x_n$ are coordinates of the Lie algebra of a maximal torus of G.

Moreover, if $f : X \to Y$ is a proper $G-$equivariant map, we can define push-forward maps $f_* : H^*_G(X) \to H^*_G(Y)$ (which are homomorphisms of $H^*_G(pt)-$modules) with the following properties:

- *projection formula:*

$$f_*(c \cdot f^* d) = f_* c \cdot d \tag{3.1}$$

- *base change:*   Suppose we have a fiber square

$$
\begin{array}{ccc}
Y \times_T Z & \xrightarrow{\ p\ } & Y \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\
Z & \xrightarrow{\ s\ } & T
\end{array}
$$

  such that the push-forward maps r_* and q_* are defined. If this happens, then we have the equality

$$s^* r_* = q_* p^* \tag{3.2}$$

- *push-forward of inclusions:*   If $i : X \hookrightarrow Y$ is an inclusion, then $i_* 1 = [X]$, where $[X]$ is the Poincare dual of the class of the subvariety X in Y. This and (3.1) imply that

$$i_* i^* c = c \cdot [X] \tag{3.3}$$

  for any $c \in H^*_G(Y)$. Moreover, for any $c \in H^*_G(X)$, [1] tells us that

$$i^* i_* c = c \cdot e(N_{Y|X}) \tag{3.4}$$

  In the above, $e(N_{Y|X})$ denotes the top Chern class of the normal bundle of X in Y.

When $\pi : X \to pt$ is the projection to a point, then the push-forward is just the integral:

$$\pi_* \alpha = \int_X \alpha$$

3.3. We will be interested in the case when $G = T = (\mathbb{C}^*)^k$ is a torus, and the fixed locus X^T of the $T-$action on X is a finite union of fixed points. In that case, we have the following *localization theorem*:

Theorem 3.5. *The restriction map*

$$H := H_T^*(X) \bigotimes_{H_T^*(pt)} \mathrm{Frac}(H_T^*(pt)) \xrightarrow{\cong} H_T^*(X^T) \bigotimes_{H_T^*(pt)} \mathrm{Frac}(H_T^*(pt))$$

is an isomorphism. For $p \in X^T$, let $i_p : pt \to X$ denote the inclusion of p in X. Then the classes

$$[p] := i_{p_*}1$$

form a basis of H as a vector space over $\mathrm{Frac}(H_T^*(pt))$.

This basis is very convenient, because $i_p^*[q] = 0$ for $p \neq q$. Moreover, by (3.4) we have

$$i_p^*[p] = e(\mathcal{T}_p X) = \prod_{w \in \mathcal{T}_p X} w \tag{3.6}$$

where $w \in \mathcal{T}_p X$ are the weights of the $T-$action in the tangent space at p. This means that if we want to compute a certain class $c \in H_T^*(X)$ in terms of the basis vectors $[p]$, all we have to do is to compute its restrictions to the fixed points. In other words, we have the following *localization formula*:

Theorem 3.7. *For any $c \in H_T^*(X)$ we have*

$$c = \sum_{p \in X^T} [p] \cdot \frac{i_p^* c}{\displaystyle\prod_{w \in \mathcal{T}_p X} w}$$

Applying π_* to the above formula, where $\pi : X \to pt$, gives us the *integral formula*:

Corollary 3.8. *For any $c \in H_T^*(X)$ we have*

$$\int_X c = \pi_* c = \sum_{p \in X^T} \frac{i_p^* c}{\displaystyle\prod_{w \in \mathcal{T}_p X} w}$$

We have already noted above the importance of the top Chern class e. In fact, we also have the equivariant version of Proposition 12.8 in [8]:

Proposition 3.9. *Suppose V is a $T-$vector bundle on X which possesses a regular section with zero locus $Z \subset X$. If the decomposition of Z into irreducible components is $Z = \cup_i Z_i$, then*

$$e(V) = \sum_i [Z_i].$$

4. LAUMON QUASIFLAG SPACES

4.1. Recall that we have chosen a basis $(w_1, ..., w_n)$ of $\mathbb{C}^n$. Let $T \subset SL_n$ be the maximal torus of matrices which are diagonal in this basis. Let $\mathfrak{t}$ be the Lie algebra of T, and let $\mathfrak{h} = \mathfrak{t}^*$ be its dual. Whenever we will mention roots and weights from now on, we will always refer to the roots and weights of $\mathfrak{h}$ (as in Section 2). For any $\gamma = -d_1\alpha_1 - ... - d_{n-1}\alpha_{n-1} \in Q^-$, let $\mathcal{M}_\gamma$ denote the moduli space of Laumon quasiflags

$$\mathcal{F} : 0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_{n-1} \subset \mathcal{O}^n \tag{4.1}$$

In the above, $\mathcal{O}$ stands for $\mathcal{O}_{\mathbb{P}^1}$, and each $\mathcal{F}_i$ is a torsion-free sheaf on $\mathbb{P}^1$ of rank i and degree $-d_i$, such that $\mathcal{F}_i|_\infty = \mathrm{span}(w_1, ..., w_i)$. Fixing the behavior of our flags at ∞ is called *framing*. As in [6], the spaces $\mathcal{M}_\gamma$ are smooth and of dimension $2(d_1 + ... + d_{n-1})$. We can define the disjoint union:

$$\mathcal{M} = \bigsqcup_{\gamma \in Q^-} \mathcal{M}_\gamma. \tag{4.2}$$

4.2. The group $T \times \mathbb{C}^*$ acts on $\mathcal{M}_\gamma$ in the following way. The torus T acts on the ambient sheaf $\mathcal{O}^n$ by changes of basis and $\mathbb{C}^*$ acts on the sheaves by multiplying the base $\mathbb{P}^1$. The fixed points of $\mathcal{M}_\gamma$ under this torus action are precisely flags which have

$$\mathcal{F}_i = w_1 \cdot \mathcal{O}(-d_i^1) \oplus ... \oplus w_i \cdot \mathcal{O}(-d_i^i) \tag{4.3}$$

where $\mathcal{O}(-1)$ will always denote $\mathcal{O}_{\mathbb{P}^1}(-1 \cdot 0_{\mathbb{P}^1})$. Thus a fixed flag is determined by a vector of non-negative integers

$$d = (d_j^i), \quad 1 \le i \le j \le n - 1 \tag{4.4}$$

satisfying

$$d_j^i \ge d_{j+1}^i, \quad d_j^1 + d_j^2 + ... + d_j^j = d_j, \quad \forall\, 1 \le i \le j \le n - 1 \tag{4.5}$$

We will always write $d_n^i = 0$ by convention. Since the vector d completely determines the fixed flag (4.3), we will often abuse notation and write d for the flag as well.

4.3. Let $a_1, ..., a_n$ be standard coordinates on $\mathbb{C}^n$, and x be the standard coordinate on $\mathbb{C}$. Then $a_1 - a_2, ..., a_{n-1} - a_n, x$ will be coordinates on $\mathrm{Lie}(T \times \mathbb{C}^*) = \mathfrak{t} \oplus \mathbb{C}$. As mentioned in Section 3.2, in this case we have

$$H^*_{T \times \mathbb{C}^*}(pt) = \mathbb{C}[a_1 - a_2, ..., a_{n-1} - a_n, x] \tag{4.6}$$

Consider

$$H := \bigoplus_{\gamma \in Q^-} H^*_{T \times \mathbb{C}^*}(\mathcal{M}_\gamma) \bigotimes_{H^*_{T \times \mathbb{C}^*}(pt)} \mathrm{Frac}(H^*_{T \times \mathbb{C}^*}(pt)) \tag{4.7}$$

By the description of the torus fixed points in (4.3), one sees that each $\mathcal{M}_\gamma$ contains finitely many fixed points. Therefore the localization Theorem 3.5 states that H is a vector space over $\mathrm{Frac}(H^*_{T \times \mathbb{C}^*}(pt))$, with a basis given by the cohomology classes of the torus fixed points. Since these torus fixed points are indexed by vectors $d = (d_j^i)$ as in (4.4), we will denote their classes in H by the symbols $[d]$.

4.4. In [15], Finkelberg and Kuznetsov introduced a geometric structure of an $\mathfrak{sl}_n$−module on H, which we will describe below. Given two flags $\mathcal{F}, \mathcal{F}' \in \mathcal{M}$, we will write $\mathcal{F}' \subset_i \mathcal{F}$ if $\mathcal{F}'_j = \mathcal{F}_j$ for $j \neq i$, while $\mathcal{F}'_i \subset \mathcal{F}_i$ in such a way that $\text{length}(\mathcal{F}_i/\mathcal{F}'_i) = 1$. Then for $1 \leq i \leq n-1$, let

$$\mathfrak{C}_i = \{(\mathcal{F}, \mathcal{F}') \in \mathcal{M} \times \mathcal{M} | \mathcal{F}' \subset_i \mathcal{F}\} \tag{4.8}$$

As in [6], we will call $\mathfrak{C}_i$ a *simple correspondence*. It is a smooth subvariety of $\mathcal{M} \times \mathcal{M}$ which will have middle dimension $2(d_1 + ... + d_{n-1}) + 1$ inside the component $\mathcal{M}_\gamma \times \mathcal{M}_{\gamma-\alpha_i}$.

Let $p, q : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be the standard projections onto the first and second factors. Define the operators $e_i, f_i : H \to H$ by

$$e_i(\alpha) = p_*([\mathfrak{C}_i] \cdot q^*\alpha)$$

$$f_i(\alpha) = -q_*([\mathfrak{C}_i] \cdot p^*\alpha)$$

If we let $\tilde{p}, \tilde{q} : \mathfrak{C}_i \to \mathcal{M}$ be the projections restricted to $\mathfrak{C}_i$, then the above definitions are equivalent to $e_i = \tilde{p}_*\tilde{q}^*$ and $f_i = -\tilde{q}_*\tilde{p}^*$. Theorem 3.4 in [4] gives us the following result:

Theorem 4.9. *The operators e_i and f_i generate a well-defined $\mathfrak{sl}_n$−action on H. Under this action, H is isomorphic to the Verma module $M(\frac{a}{x} - \rho)$.*

Remark 4.10. The slight difference between the above theorem and its counterpart in [4] is due to the fact that we use highest weight Verma modules instead of lowest weight Verma modules, but the theory is analogous.

Let $\gamma = -d_1\alpha_1 - ... - d_{n-1}\alpha_{n-1}$, and consider the cohomology class $[d] \in H$ corresponding to a fixed point $d \in \mathcal{M}_\gamma^{T \times \mathbb{C}^*}$. We want to write down how the Cartan elements $h_i = [e_i, f_i]$ act on $[d]$. Section 3.7 of [15] tells us that

$$h_i([d]) = \left(\frac{a_i}{x} - \frac{a_{i+1}}{x} - 1 + d_{i-1} - 2d_i + d_{i+1}\right) \cdot [d] = \left\langle h_i, \frac{a}{x} - \rho + \gamma \right\rangle \cdot [d] \tag{4.11}$$

Therefore the weight spaces of the $\mathfrak{sl}_n$−module H are precisely the summands of (4.7), namely

$$H\left[\frac{a}{x} - \rho + \gamma\right] = H_{T \times \mathbb{C}^*}^*(\mathcal{M}_\gamma) \bigotimes_{H_{T \times \mathbb{C}^*}^*(pt)} \text{Frac}(H_{T \times \mathbb{C}^*}^*(pt)). \tag{4.12}$$

4.5. One can generalize the definition of the simple correspondences $\mathfrak{C}_i$ to obtain even more operators on H, with the goal of studying the locus

$$\mathfrak{C} = \{(\mathcal{F}, \mathcal{F}') \in \mathcal{M} \times \mathcal{M} | \mathcal{F}' \subset \mathcal{F}\} \tag{4.13}$$

The notation $\mathcal{F}' \subset \mathcal{F}$ means that $\mathcal{F}'_i \subset \mathcal{F}_i$ for all $1 \leq i \leq n-1$, without any extra conditions. For such $\mathcal{F}' \subset \mathcal{F}$, the "quotient" $\mathcal{F}/\mathcal{F}'$ should be interpreted as the flag of quotients

$$\mathcal{F}_1/\mathcal{F}'_1 \to ... \to \mathcal{F}_{n-1}/\mathcal{F}'_{n-1} \tag{4.14}$$

In the above, the maps are induced by the flag inclusion maps. Because $\mathcal{F}_i$ and $\mathcal{F}_i'$ have the same rank for all i, the quotient $\mathcal{F}/\mathcal{F}'$ is supported at finitely many points:

$$\operatorname{supp}(\mathcal{F}/\mathcal{F}') = \sum_{s \in \mathbb{P}^1} \alpha_s \cdot s$$

For each $s \in \mathbb{P}^1$ that appears in the above sum, α_s is a vector of natural numbers $(\alpha_s^{(1)}, ..., \alpha_s^{(n-1)})$ such that $\alpha_s^{(i)}$ is the length of the quotient sheaf $\mathcal{F}_i/\mathcal{F}_i'$ at s.

The stalk of $\mathcal{F}/\mathcal{F}'$ at any given point $s \in \mathbb{P}^1$ is a flag of finite dimensional vector spaces (with maps given by (4.14)), and is thus a representation of the quiver A_{n-1} of Section 2.5. Let

$$\gamma_s \in \operatorname{Rep}(A_{n-1})$$

denote the isomorphism class of this representation. We know from Section 2.5 that any isomorphism class $\kappa \in \operatorname{Rep}(A_{n-1})$ can be uniquely written as a direct sum $\kappa = \oplus_{j=1}^m \theta_j$, where the θ_j are classes of indecomposable representations. Then we define the locus

$$\mathfrak{C}_\kappa^\circ \subset \mathfrak{C}$$

to be the set of $(\mathcal{F}, \mathcal{F}')$ with $\mathcal{F}' \subset \mathcal{F}$ which satisfy the conditions:

(1) there exist distinct points $s_1, ..., s_m \in \mathbb{P}^1$ such that

$$\operatorname{supp}(\mathcal{F}/\mathcal{F}') = \sum_{j=1}^m \dim(\theta_j) \cdot s_j$$

(2)
$$\gamma_{s_j} = \theta_j$$

Finally, we will call the closure $\mathfrak{C}_\kappa = \overline{\mathfrak{C}_\kappa^\circ} \subset \mathfrak{C}$ a *correspondence*. Note that when κ is the simple representation corresponding to the vertex i of the quiver, then $\mathfrak{C}_\kappa = \mathfrak{C}_i$ as defined in the previous subsection. In [15] we encounter the following result:

Proposition 4.15. *The correspondences $\mathfrak{C}_\kappa$ are precisely the distinct irreducible components of $\mathfrak{C}$ of maximal dimension, and all have dimension equal to half the dimension of $\mathcal{M} \times \mathcal{M}$.*

Since $\mathcal{M} \times \mathcal{M}$ is a disjoint union of components of various dimensions, the above statement needs a few clarifications. The correspondence $\mathfrak{C}_\kappa$ intersects the component $\mathcal{M}_\gamma \times \mathcal{M}_{\gamma'}$ only if $\gamma' - \gamma = \dim(\kappa)$. What the proposition says is that in each component $\mathcal{M}_\gamma \times \mathcal{M}_{\gamma'}$, the subvarieties $\mathfrak{C}_\kappa$ with $\gamma' - \gamma = \dim(\kappa)$ are precisely the distinct irreducible components of $\mathfrak{C}$ of maximal dimension. Moreover, the dimension of $\mathfrak{C}_\kappa$ in the component $\mathcal{M}_\gamma \times \mathcal{M}_{\gamma'}$ is equal to half the dimension of $\mathcal{M}_\gamma \times \mathcal{M}_{\gamma'}$.

The correspondences $\mathfrak{C}_\kappa$ define operators on H via

$$e_\kappa(\alpha) = p_*([\mathfrak{C}_\kappa] \cdot q^* \alpha)$$

where $p, q : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ are the standard projections onto the first and second factors, respectively. To understand how the operators e_κ act, we have the following:

Theorem 4.16. *The map that sends the elements $e_\kappa \in U(A_{n-1})^{op}$ to the operators e_κ defined above is an algebra homomorphism. Here $U(A_{n-1})^{op}$ means the algebra opposite to the Ringel-Hall algebra of A_{n-1}.*

In other words, the operators e_κ follow the same multiplication rules as the corresponding symbols from the opposite Ringel-Hall algebra of A_{n-1}. We will not prove Theorem 4.16 here, since it has been proved in the more general case of parabolic flags of sheaves in [16], Theorem 7.12. Note that in [16] the theorem is proved for the operators given by the transposed correspondences $\mathfrak{C}_\kappa^T$. Though the proof is completely analogous, this accounts for the word "opposite" in the statement of Theorem 4.16.

5. The Vector Bundle E

5.1. Let us consider the sheaf E on $\mathcal{M} \times \mathcal{M}$, whose fiber over $(\mathcal{F}, \mathcal{F}') \in \mathcal{M} \times \mathcal{M}$ is

$$E|_{(\mathcal{F}, \mathcal{F}')} = \mathrm{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F}) \tag{5.1}$$

By definition, elements of $\mathrm{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F})$ are tuples $(\phi_1, ..., \phi_{n-1})$ of morphisms $\phi_i : \mathcal{F}_i'(\infty) \to \mathcal{O}^n/\mathcal{F}_i$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{F}_1'(\infty) & \xrightarrow{\ \phi_1\ } & \mathcal{O}^n/\mathcal{F}_1 \\
\downarrow & & \downarrow \\
\mathcal{F}_2'(\infty) & \xrightarrow{\ \phi_2\ } & \mathcal{O}^n/\mathcal{F}_2 \\
\downarrow & & \downarrow \\
... & \longrightarrow & ... \\
\downarrow & & \downarrow \\
\mathcal{F}_{n-1}'(\infty) & \xrightarrow{\ \phi_{n-1}\ } & \mathcal{O}^n/\mathcal{F}_{n-1}
\end{array}
\tag{5.2}
$$

commutes. In the above, all the vertical maps are induced by the flag inclusions.

Proposition 5.3. *The restriction of E to the diagonal $\Delta \subset \mathcal{M} \times \mathcal{M}$ is just the tangent bundle of $\mathcal{M}$.*

Proof The proposition is merely a consequence of the fact that the tangent space to $\mathcal{M}$ at $\mathcal{F}$ is $\mathrm{Hom}(\mathcal{F}(\infty), \mathcal{O}^n/\mathcal{F})$, as described in [2] and [11]. The extra torsion by $\infty \in \mathbb{P}^1$ is a consequence of the fact that our flags are fixed at that point, and thus there cannot be any deformation of their fibers at $\infty \in \mathbb{P}^1$. □

5.2. The sheaf E admits an alternative description, which will be very useful in establishing certain technical results later on. Let $\pi : \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1$ be the standard projection, and denote $L_0 = \mathbb{P}^1 \times 0$. Following [3], to any flag of sheaves as in (4.1) we associate the torsion-free subsheaf $\tilde{\mathcal{F}} \subset \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}$ given by

$$\tilde{\mathcal{F}} = \pi^* \mathcal{F}_1 + \pi^* \mathcal{F}_2(-L_0)... + \pi^* \mathcal{F}_{n-1}(-(n-2)L_0) + \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}(-(n-1)L_0)$$

The sum does not refer to a direct sum, but to a sum of subsheaves of $\mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}$. Constants $z \in \mathbb{C}^*$ act on $\mathbb{P}^1 \times \mathbb{A}^1$ by multiplying the second factor, and under this action each summand $\pi^* \mathcal{F}_{i+1}(-iL_0)$ of $\tilde{\mathcal{F}}$ is preserved. Therefore, $\tilde{\mathcal{F}}$ is a $z-$invariant subsheaf of $\mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}$.

5.3. Let $D_\infty = \infty \times \mathbb{A}^1$. Define $\mathrm{Hom}_z(\tilde{\mathcal{F}}'(D_\infty), \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\tilde{\mathcal{F}})$ to be the space of $z-$invariant morphisms between the sheaves in question, and we will show that there is a natural isomorphism

$$\mathrm{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F}) \cong \mathrm{Hom}_z(\tilde{\mathcal{F}}'(D_\infty), \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\tilde{\mathcal{F}}). \qquad (5.4)$$

To define this isomorphism, take an element $\phi \in \mathrm{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F})$ as in (5.2). Each one of its component morphisms

$$\phi_{i+1} : \mathcal{F}'_{i+1}(\infty) \to \mathcal{O}^n/\mathcal{F}_{i+1}$$

naturally lifts to a $z-$invariant morphism

$$\tilde{\phi}'_{i+1} : \pi^* \mathcal{F}'_{i+1}(D_\infty) \to \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\pi^* \mathcal{F}_{i+1}$$

Let us twist this morphism by $-iL_0$, and then compose it with the natural map

$$\mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}(-iL_0)/\pi^* \mathcal{F}_{i+1}(-iL_0) \hookrightarrow \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\pi^* \mathcal{F}_{i+1}(-iL_0) \twoheadrightarrow \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\tilde{\mathcal{F}}$$

In this way we obtain $z-$invariant homomorphisms

$$\tilde{\phi}_{i+1} : \pi^* \mathcal{F}'_{i+1}(D_\infty - iL_0) \to \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\tilde{\mathcal{F}}$$

By the commutativity of the diagram (5.2), these homomorphisms paste to give a well-defined $z-$invariant morphism $\tilde{\phi} : \tilde{\mathcal{F}}'(D_\infty) \to \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\tilde{\mathcal{F}}$.

Conversely, take $\tilde{\phi} \in \mathrm{Hom}_z(\tilde{\mathcal{F}}'(D_\infty), \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\tilde{\mathcal{F}})$. By restriction, it gives rise to $z-$invariant homomorphisms

$$\tilde{\phi}_{i+1} : \pi^* \mathcal{F}'_{i+1}(D_\infty - iL_0) \to \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\tilde{\mathcal{F}}$$

For each i, the fact that this morphism is $z-$invariant implies that it lifts to a morphism into $\mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\pi^* \mathcal{F}_{i+1}(-iL_0)$, and moreover that its image lies in $\mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}(-iL_0)/\pi^* \mathcal{F}_{i+1}(-iL_0)$. Twisting by iL_0 gives rise to a $z-$invariant homomorphism

$$\tilde{\phi}'_{i+1} : \pi^* \mathcal{F}'_{i+1}(D_\infty) \to \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\pi^* \mathcal{F}_{i+1}$$

Since all the sheaves in the above are pulled back from $\mathbb{P}^1$, then Lemma 5.10 below implies that $\tilde{\phi}'_{i+1}$ is the lift of a homomorphism of sheaves on $\mathbb{P}^1$

$$\phi_{i+1} : \mathcal{F}'_{i+1}(\infty) \to \mathcal{O}^n/\mathcal{F}_{i+1}$$

The fact that the homomorphisms $\tilde{\phi}_{i+1}$ paste to give the homomorphism $\tilde{\phi}$ implies precisely that the morphisms ϕ_{i+1} make the diagram (5.2) commute. Therefore they give rise to an element $\phi \in \mathrm{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F})$.

It is easily seen that the procedures $\phi \to \tilde{\phi}$ and $\tilde{\phi} \to \phi$ described above are inverse to each other, and thus they give the desired natural isomorphism (5.4). Thus we may conclude that E is the sheaf whose fiber above $(\mathcal{F}, \mathcal{F}')$ is

$$E|_{(\mathcal{F}, \mathcal{F}')} = \mathrm{Hom}_z(\tilde{\mathcal{F}}'(D_\infty), \mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{A}^1}/\tilde{\mathcal{F}}). \qquad (5.5)$$

5.4. For any $i \geq 2$ and any coherent sheaves $\mathcal{S}, \mathcal{S}'$ on $\mathbb{P}^1 \times \mathbb{A}^1$, we have $\mathrm{Ext}^i(\mathcal{S}, \mathcal{S}') = 0$. This can be easily seen by covering $\mathbb{P}^1 \times \mathbb{A}^1$ with the affine charts $(\mathbb{P}^1 - 0) \times \mathbb{A}^1$ and $(\mathbb{P}^1 - \infty) \times \mathbb{A}^1$, and then using the Čech complex to compute $\mathrm{Ext}^i(\mathcal{S}, \mathcal{S}')$. Then let us define the "virtual vector space"

$$\chi(\mathcal{S}, \mathcal{S}') = \mathrm{Hom}(\mathcal{S}, \mathcal{S}') - \mathrm{Ext}^1(\mathcal{S}, \mathcal{S}') \tag{5.6}$$

If $\mathcal{S}$ and $\mathcal{S}'$ are $z-$invariant sheaves on $\mathbb{P}^1 \times \mathbb{A}^1$, we can take the $z-$invariant parts of the Ext groups in question, and define

$$\chi_z(\mathcal{S}, \mathcal{S}') = \mathrm{Hom}_z(\mathcal{S}, \mathcal{S}') - \mathrm{Ext}^1_z(\mathcal{S}, \mathcal{S}') \tag{5.7}$$

The advantage of the functor χ over the functors Ext^i is the following additivity property: suppose $\mathcal{S}, \mathcal{S}', \mathcal{S}'', \mathcal{S}_0$ are coherent sheaves such that the sequence

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$$

is exact. Then we have the following relations in the Grothendieck group:

$$\chi(\mathcal{S}_0, \mathcal{S}) = \chi(\mathcal{S}_0, \mathcal{S}') + \chi(\mathcal{S}_0, \mathcal{S}'')$$

$$\chi(\mathcal{S}, \mathcal{S}_0) = \chi(\mathcal{S}', \mathcal{S}_0) + \chi(\mathcal{S}'', \mathcal{S}_0) \tag{5.8}$$

The same additivity property holds for the functor χ_z, when $\mathcal{S}, \mathcal{S}', \mathcal{S}'', \mathcal{S}_0$ are $z-$invariant sheaves. Moreover, we have

$$\chi(\mathcal{S} \otimes L, \mathcal{S}' \otimes L) = \chi(\mathcal{S}, \mathcal{S}') \tag{5.9}$$

for any coherent sheaves $\mathcal{S}, \mathcal{S}'$ and line bundle L. The same property holds for χ_z, when $\mathcal{S}, \mathcal{S}'$ and L are $z-$invariant.

5.5. The above natural properties of the functors χ and χ_z will allow us to reinterpret the sheaf E. To do this, we will need the following technical result:

Lemma 5.10. *Let $\mathcal{S}, \mathcal{S}'$ be locally free sheaves on $\mathbb{P}^1$. Then for every $i \geq 0$ we have natural isomorphisms*

$$\mathrm{Ext}^i_z(\pi^*\mathcal{S}'(kL_0), \pi^*\mathcal{S}) \cong \mathrm{Ext}^i(\mathcal{S}', \mathcal{S}), \text{ if } k \leq 0 \tag{5.11}$$

$$\mathrm{Ext}^i_z(\pi^*\mathcal{S}'(kL_0), \pi^*\mathcal{S}) \cong 0, \text{ if } k > 0 \tag{5.12}$$

Proof It is enough to prove the lemma for $i = 0$, namely for the functor Hom. This is because the same argument will prove the Lemma for arbitrary i, by using the Čech complexes with respect to the affine coverings

$$\mathbb{P}^1 \times \mathbb{A}^1 = (\mathbb{P}^1 - 0) \times \mathbb{A}^1 \cup (\mathbb{P}^1 - \infty) \times \mathbb{A}^1, \quad \mathbb{P}^1 = (\mathbb{P}^1 - 0) \cup (\mathbb{P}^1 - \infty)$$

to compute the functors Ext^i_z and Ext^i, respectively.

So let us describe a $z-$invariant homomorphism from $\pi^*\mathcal{S}'(kL_0)$ to $\pi^*\mathcal{S}$. The local sections of $\pi^*\mathcal{S}'(kL_0)$ are generated over z by expressions of the form $P(x)z^{-k}$, where x is a local coordinate on $\mathbb{P}^1$ and $P(x)$ is a local section of $\mathcal{S}'$. If we want to map this in a $z-$invariant way, we must send it to some expression of the form $Q(x)z^{-k}$. But if $k > 0$, there simply are no such local sections in $\pi^*\mathcal{S}$, which proves (5.12).

If, on the other hand, $k \leq 0$, then any $z-$invariant morphism is completely determined by sending local sections $P(x)z^{-k}$ of $\pi^*\mathcal{S}'(kL_0)$ to local sections $Q(x)z^{-k}$ of $\pi^*\mathcal{S}$. This amounts to sending local sections $P(x)$ of $\mathcal{S}'$ to local sections $Q(x)$ of $\mathcal{S}$. But this is just the data that defines a homomorphism from $\mathcal{S}'$ to $\mathcal{S}$, and this proves (5.11).

$\square$

This lemma has a very useful corollary. Consider the following resolution of $\tilde{\mathcal{F}}$:

$$0 \to \bigoplus_{i=1}^{n-1} \pi^*\mathcal{F}_i(-iL_0) \to \bigoplus_{i=1}^{n} \pi^*\mathcal{F}_i(-(i-1)L_0) \to \tilde{\mathcal{F}} \to 0 \qquad (5.13)$$

where we write $\mathcal{F}_n = \mathcal{O}^n$ by convention. The second non-zero arrow sends the direct sum to the sum of sheaves, while the first non-zero arrow sends the summand $\pi^*\mathcal{F}_i(-iL_0)$ into $\pi^*\mathcal{F}_i(-(i-1)L_0)$ via the standard embedding $\mathcal{O}(-L_0) \hookrightarrow \mathcal{O}$ and into $\pi^*\mathcal{F}_{i+1}(-iL_0)$ via minus the embedding $\mathcal{F}_i \hookrightarrow \mathcal{F}_{i+1}$.

Lemma 5.14. *For any flag $\mathcal{F} \in \mathcal{M}$ and any $i \geq 0$, we have*

$$\mathrm{Ext}_z^i(\tilde{\mathcal{F}}(D_\infty), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1}) = 0$$

Proof Obviously, it is enough to prove this result for $i = 0, 1$. For $i = 0$, in a similar way with (5.4) one shows that

$$\mathrm{Hom}_z(\tilde{\mathcal{F}}(D_\infty), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1}) \cong \mathrm{Hom}(\mathcal{F}(\infty), \mathcal{O})$$

By definition, the space in the right hand side consists of commutative diagrams of morphisms of sheaves on $\mathbb{P}^1$:

$$
\begin{array}{ccc}
\mathcal{F}_1(\infty) & \xrightarrow{\phi_1} & \mathcal{O} \\
\downarrow & & \downarrow \\
\ldots & \longrightarrow & \ldots \\
\downarrow & & \downarrow \\
\mathcal{F}_{n-1}(\infty) & \xrightarrow{\phi_{n-1}} & \mathcal{O} \\
\downarrow & & \downarrow \\
\mathcal{O}(\infty) & \xrightarrow{\phi_n} & \mathcal{O}
\end{array}
$$

From the above commutative diagram, we see that all the maps $\phi_1,...,\phi_{n-1}$ are determined by ϕ_n via restriction. But because of the torsion with ∞, we must have $\phi_n = 0$. Thus, there are no non-trivial elements of $\mathrm{Hom}(\mathcal{F}(\infty), \mathcal{O})$, and this proves the lemma for $i = 0$.

Now apply relation (5.8) to the short exact sequence (5.13) (twisted by D_∞). We obtain

$$\chi_z(\tilde{\mathcal{F}}(D_\infty), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1}) = \sum_{k=1}^{n} \chi_z(\pi^*\mathcal{F}_k(D_\infty - (k-1)L_0), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1}) -$$

$$- \sum_{k=1}^{n-1} \chi_z(\pi^*\mathcal{F}_k(D_\infty - kL_0), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1}) \qquad (5.15)$$

To prove the Lemma for $i = 1$, it is enough to show that $\text{Ext}_z^1(\tilde{\mathcal{F}}(D_\infty), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1})$ has dimension 0. For this, it is enough to show that the virtual vector space $\chi_z(\tilde{\mathcal{F}}(D_\infty), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1})$ has dimension 0. Thus it is enough to show that the two sums of vector spaces in (5.15) have the same dimension. But from Lemma 5.10 we have

$$\sum_{k=1}^n \chi_z(\pi^* \mathcal{F}_k(D_\infty - (k-1)L_0), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1}) = \sum_{k=1}^n \chi(\mathcal{F}_k(\infty), \mathcal{O})$$

$$\sum_{k=1}^{n-1} \chi_z(\pi^* \mathcal{F}_k(D_\infty - kL_0), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{A}^1}) = \sum_{k=1}^{n-1} \chi(\mathcal{F}_k(\infty), \mathcal{O})$$

Since $\chi(\mathcal{O}^n(\infty), \mathcal{O}) = 0$, the above sums are equal, and thus they have the same dimension. This concludes the proof. $\qquad\square$

5.6. Now to reinterpret E. The short exact sequence

$$0 \to \tilde{\mathcal{F}} \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^n \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^n / \tilde{\mathcal{F}} \to 0$$

induces the long exact sequence of Ext_z groups:

$$\text{Hom}_z(\tilde{\mathcal{F}}'(D_\infty), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^n) \to \text{Hom}_z(\tilde{\mathcal{F}}'(D_\infty), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^n / \tilde{\mathcal{F}}) \to$$

$$\to \text{Ext}_z^1(\tilde{\mathcal{F}}'(D_\infty), \tilde{\mathcal{F}}) \to \text{Ext}_z^1(\tilde{\mathcal{F}}'(D_\infty), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^n) \qquad (5.16)$$

Lemma 5.14 implies that the vector spaces at the ends of the sequence are 0, and therefore

$$\text{Hom}_z(\tilde{\mathcal{F}}'(D_\infty), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^n / \tilde{\mathcal{F}}) \cong \text{Ext}_z^1(\tilde{\mathcal{F}}'(D_\infty), \tilde{\mathcal{F}}) \qquad (5.17)$$

However, we have that $\text{Hom}_z(\tilde{\mathcal{F}}'(D_\infty), \tilde{\mathcal{F}}) = 0$. This is proven word by word as the $i = 0$ statement of Lemma 5.14, so we will not repeat the proof here. Therefore, relations (5.17) and (5.5) imply that E is the sheaf with fiber over $(\mathcal{F}, \mathcal{F}')$:

$$E|_{(\mathcal{F}, \mathcal{F}')} = -\chi_z(\tilde{\mathcal{F}}'(D_\infty), \tilde{\mathcal{F}}). \qquad (5.18)$$

From Corollary 7.9.9. of [18], we know that the dimension of the virtual vector spaces $\chi_z(\tilde{\mathcal{F}}'(D_\infty), \tilde{\mathcal{F}})$ is a locally constant function of $\mathcal{F}$ and $\mathcal{F}'$. Therefore, so is the dimension of the fibers of E. This implies that E is a vector bundle, and its rank can be computed from the character of $T \times \mathbb{C}^*$ in the fixed fibers of E (just set $x, a_1, ..., a_n \to 0$ in Proposition 5.22). Summarizing everything, we have:

Proposition 5.19. *The sheaf E is a vector bundle on $\mathcal{M} \times \mathcal{M}$, of rank equal to half the dimension of the base $\mathcal{M} \times \mathcal{M}$.*

5.7. We will use the resolutions (5.13) for $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ to compute the $K-$theoretic class of E. Properties (5.8) and (5.9) of the functor χ imply the following relations in the Grothendieck group of vector spaces:

$$E|_{(\mathcal{F}, \mathcal{F}')} = -\chi_z(\tilde{\mathcal{F}}'(D_\infty), \tilde{\mathcal{F}}) =$$

$$= -\sum_{i=1}^n \sum_{j=1}^n \chi_z(\pi^* \mathcal{F}_i'(D_\infty + (j-i)L_0), \pi^* \mathcal{F}_j) + \sum_{i=1}^n \sum_{j=1}^{n-1} \chi_z(\pi^* \mathcal{F}_i'(D_\infty + (j-i+1)L_0), \pi^* \mathcal{F}_j) +$$

$$+ \sum_{i=1}^{n-1} \sum_{j=1}^n \chi_z(\pi^* \mathcal{F}_i'(D_\infty + (j-i-1)L_0), \pi^* \mathcal{F}_j) - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \chi_z(\pi^* \mathcal{F}_i'(D_\infty + (j-i)L_0), \pi^* \mathcal{F}_j)$$

Lemma 5.10 allows us to compute the above spaces, yielding

$$E|_{(\mathcal{F},\mathcal{F}')} = -\sum_{j\leq i\leq n}\chi(\mathcal{F}_i'(\infty),\mathcal{F}_j) + \sum_{j\leq i-1\leq n-1}\chi(\mathcal{F}_i'(\infty),\mathcal{F}_j) + \sum_{j\leq i+1\leq n}\chi(\mathcal{F}_i'(\infty),\mathcal{F}_j) -$$

$$-\sum_{j\leq i\leq n-1}\chi(\mathcal{F}_i'(\infty),\mathcal{F}_j) = \sum_{i=1}^{n-1}\chi(\mathcal{F}_i'(\infty),\mathcal{F}_{i+1}) - \sum_{i=1}^{n-1}\chi(\mathcal{F}_i'(\infty),\mathcal{F}_i) \tag{5.20}$$

Since all the identifications we have made along the way are natural, the above equalities paste over all $(\mathcal{F},\mathcal{F}')$ to give the following equality in the Grothendieck group of coherent sheaves on $\mathcal{M}\times\mathcal{M}$:

$$E = \sum_{i=1}^{n-1}G_i - \sum_{i=1}^{n-1}H_i \tag{5.21}$$

In the above, G_i and H_i are the sheaves whose fibers over $(\mathcal{F},\mathcal{F}')$ are $\chi(\mathcal{F}_i'(\infty),\mathcal{F}_{i+1})$ and $\chi(\mathcal{F}_i'(\infty),\mathcal{F}_i)$, respectively. By Corollary 7.9.9. of [18], they are vector bundles.

5.8. Relation (5.21) allows us to compute the character of $T\times\mathbb{C}^*$ in the fixed fibers of E. Knowing the character implies knowing the weights of the $T\times\mathbb{C}^*$ action, since the character is just the sum of the exponentials of the weights. But knowing the weights implies knowing the equivariant Chern classes of E, by Theorem 3.7. Therefore, let $\mathcal{F},\mathcal{F}'$ be fixed points of $\mathcal{M}$ corresponding to the vectors of non-negative integers d,d' as in (4.4), and let us denote the fiber of E above $(\mathcal{F},\mathcal{F}')$ by $E_{d,d'}$.

Proposition 5.22. *With the above notations, the character* $\mathrm{char}(E_{d,d'})$ *of* $T\times\mathbb{C}^*$ *in the fiber* $E_{d,d'}$ *equals*

$$\sum_{i=1}^{n-1}\sum_{j=1}^{i+1}\sum_{j'=1}^{i}\frac{e^{a_j}}{e^{a_{j'}}}\cdot\frac{e^{x(d_i'^{j'}-d_{i+1}^j+1)}-e^x}{e^x-1} - \sum_{i=1}^{n-1}\sum_{j=1}^{i}\sum_{j'=1}^{i}\frac{e^{a_j}}{e^{a_{j'}}}\cdot\frac{e^{x(d_i'^{j'}-d_i^j+1)}-e^x}{e^x-1}$$

Proof If we have a short exact sequence of representations of $T\times\mathbb{C}^*$:

$$0\to U\to V\to W\to 0$$

then it is easy to see that the character is additive:

$$\mathrm{char}(V) = \mathrm{char}(U) + \mathrm{char}(W)$$

Therefore, by (5.21), it is enough to show that

$$\mathrm{char}(G_i|_{d,d'}) = \sum_{j=1}^{i+1}\sum_{j'=1}^{i}\frac{e^{a_j}}{e^{a_{j'}}}\cdot\frac{e^{x(d_i'^{j'}-d_{i+1}^j+1)}-e^x}{e^x-1} \tag{5.23}$$

$$\mathrm{char}(H_i|_{d,d'}) = \sum_{j=1}^{i}\sum_{j'=1}^{i}\frac{e^{a_j}}{e^{a_{j'}}}\cdot\frac{e^{x(d_i'^{j'}-d_i^j+1)}-e^x}{e^x-1} \tag{5.24}$$

for each $1\leq i\leq n-1$. We will prove the first of these statements, as the second one is completely analogous. By (4.3), we have

$$\mathcal{F}_i' = w_1\cdot\mathcal{O}(-d_i'^1)\oplus...\oplus w_i\cdot\mathcal{O}(-d_i'^i)$$

$$\mathcal{F}_{i+1} = w_1 \cdot \mathcal{O}(-d_{i+1}^1) \oplus ... \oplus w_{i+1} \cdot \mathcal{O}(-d_{i+1}^{i+1})$$

By (5.8), we therefore have

$$G_i|_{d,d'} = \chi(\mathcal{F}_i'(\infty), \mathcal{F}_{i+1}) = \sum_{j=1}^{i+1} \sum_{j'=1}^{i} \chi(w_{j'} \cdot \mathcal{O}(\infty - d_i'^{j'}), w_j \cdot \mathcal{O}(-d_{i+1}^j))$$

The character of $\mathbb{C}^*$ acting in $\chi(\mathcal{O}(\infty + k), \mathcal{O}(l))$ is easily seen to be $\frac{e^{x(l-k+1)} - e^x}{e^x - 1}$.
Therefore, the character of $T \times \mathbb{C}^*$ in $\chi(w_{j'} \cdot \mathcal{O}(\infty - d_i'^{j'}), w_j \cdot \mathcal{O}(-d_{i+1}^j))$ equals

$$\frac{e^{a_j}}{e^{a_{j'}}} \cdot \frac{e^{x(d_i'^{j'} - d_{i+1}^j + 1)} - e^x}{e^x - 1}$$

Summing this up over all j, j' implies the desired relation (5.23). $\square$

6. The Operators $A(m)$

6.1. Given a $T \times \mathbb{C}^*$−vector bundle V of rank r, its *Chern polynomial* is defined as

$$c(V, t) = c_r(V) + c_{r-1}(V) \cdot t + c_{r-2}(V) \cdot t^2 + ... + c_0(V) \cdot t^r$$

where $c_0(V) = 1, c_1(V), ..., c_r(V)$ are the Chern classes of V. The top Chern class will always be denoted by $c_r(V) = e(V)$, as in Section 3. In case V is a $T \times \mathbb{C}^*$−vector bundle over a point (i.e. a representation of $T \times \mathbb{C}^*$), then we have

$$c(V, t) = \prod_{w \in V} (w + t)$$

In the above, $\prod_{w \in V}$ denotes the product over the weights of $T \times \mathbb{C}^*$ acting in V. As in Section 3.2, these weights are all elements of $H^*_{T \times \mathbb{C}^*}(pt)$.

By Proposition 5.19, the sheaf E introduced in the previous section is a vector bundle. Take a complex number m and let $x \in H^*_{T \times \mathbb{C}^*}(pt)$ be as in Section 4.3. Then we define the operator $A(m) : H \to H$ by

$$A(m)(\alpha) = p_*(c(E, mx) \cdot q^* \alpha) \tag{6.1}$$

where $p, q : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ are the standard projections onto the first and second factors, respectively.

6.2. By Theorem 3.7, we have

$$c(E, mx) \cdot q^*[d] = \sum_{d'} [d', d] \cdot \frac{c(E, mx)|_{d', d}}{\prod_{w \in \mathcal{T}_{d'} \mathcal{M}} w} = \sum_{d'} [d', d] \cdot \frac{\prod_{w \in E_{d', d}} (w + mx)}{\prod_{w \in \mathcal{T}_{d'} \mathcal{M}} w}$$

Clearly, $p_*[d', d] = [d']$ and thus

$$A(m)[d] = \sum_{d'} [d'] \cdot \frac{\prod\limits_{w \in E_{d',d}} (w + mx)}{\prod\limits_{w \in \mathcal{T}_{d'}\mathcal{M}} w} \tag{6.2}$$

Recall that the generalized character of $A(m)$ is, by the definition in Section 2.3,

$$\chi_{A(m)} = \sum_{\lambda} e^{\lambda} \cdot \operatorname{Tr}(A(m)|_{H[\lambda]}) \tag{6.3}$$

We will evaluate the above traces in the basis $[d]$ of H. Recall from (4.12) that a basis vector $[d]$ lies in the $\lambda-$weight space of H if and only if

$$\lambda = \frac{a}{x} - \rho + \gamma, \quad \text{where} \quad \gamma = -d_1\alpha_1 - ... - d_{n-1}\alpha_{n-1} \in Q^-$$

Then (6.2) implies that

$$\chi_{A(m)} = \sum_{\gamma \in Q^-} e^{\frac{a}{x} - \rho + \gamma} \cdot \sum_{d \in \mathcal{M}_\gamma^{T \times \mathbb{C}^*}} \frac{\prod\limits_{w \in E_{d,d}} (w + mx)}{\prod\limits_{w \in \mathcal{T}_d\mathcal{M}} w} =$$

By using the isomorphism $E_{d,d} \cong \mathcal{T}_d\mathcal{M}$ of Proposition 5.3, the above becomes

$$\chi_{A(m)} = \sum_{\gamma \in Q^-} e^{\frac{a}{x} - \rho + \gamma} \cdot \sum_{d \in \mathcal{M}_\gamma^{T \times \mathbb{C}^*}} \prod_{w \in \mathcal{T}_d\mathcal{M}} \frac{w + mx}{w} \tag{6.4}$$

But Corollary 3.8 gives us

$$\int_{\mathcal{M}_\gamma} c(\mathcal{T}\mathcal{M}_\gamma, mx) = \sum_{d \in \mathcal{M}_\gamma^{T \times \mathbb{C}^*}} \prod_{w \in \mathcal{T}_d\mathcal{M}} \frac{w + mx}{w} \tag{6.5}$$

Therefore relations (6.4) and (6.5) imply the following:

Proposition 6.6. *The character of $A(m)$ is related to the generating function $Z(m)$ of (1.2) by*

$$\chi_{A(m)} = \sum_{\gamma \in Q^-} e^{\frac{a}{x} - \rho + \gamma} \cdot \int_{\mathcal{M}_\gamma} c(\mathcal{T}\mathcal{M}_\gamma, mx) = e^{\frac{a}{x} - \rho} \cdot Z(m).$$

6.3. We will now seek to relate the operator $A(m)$ to the $\mathfrak{sl}_n-$action on H given by the operators e_i, f_i of Section 4.4.

Proposition 6.7. *The operator $A(0)$ belongs to the SL_n-action, and is precisely*

$$A(0) = \prod_{\theta \in R^+} \exp(e_\theta) = \begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & 1 & 1 & ... & 1 \\ 0 & 0 & 1 & ... & 1 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{pmatrix} =: g \in SL_n$$

The order of the terms in the product corresponds to the descending order of the positive roots

$$w_{n-1}-w_n > w_{n-2}-w_n > w_{n-2}-w_{n-1} > ... > w_1-w_n > w_1-w_{n-1} > ... > w_1-w_2$$

Remark 6.8. A priori, the $\mathfrak{sl}_n-$action on H doesn't immediately give a well-defined SL_n action on H, but only on a suitable completion of H. However, since H is isomorphic to the Verma module, any vector in H is annihilated by a high enough power of the e_i's. Therefore the "upper triangular" part of SL_n does act correctly on the space H.

Proof The second equality is easily proved as an equality in SL_n. Indeed, $e_{w_i-w_j}$ corresponds to the matrix $E_{i,j}$ which has entry 1 at the intersection of row i and column j, and 0 everywhere else. Then we have

$$\prod_{\theta \in R^+} \exp(e_\theta) = \prod_{i=n-1}^{1} \prod_{j=n}^{i+1} \exp(E_{i,j}) = \prod_{i=n-1}^{1} \prod_{j=n}^{i+1} (1 + E_{i,j})$$

But one easily notes that if $i > i'$ or if $i = i'$ and $j > j'$, then $E_{i,j} \cdot E_{i',j'} = 0$. This means that all products of more than one $E_{i,j}$ in the above expression vanish, and therefore

$$\prod_{\theta \in R^+} \exp(e_\theta) = 1 + \sum_{i=n-1}^{1} \sum_{j=n}^{i+1} E_{i,j} = \begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & 1 & 1 & ... & 1 \\ 0 & 0 & 1 & ... & 1 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$$

Let us now turn to the proof of the first equality in the statement of Proposition 6.7.

Lemma 6.9. *There exists a regular section of E which vanishes on the locus*

$$\mathfrak{C} = \{(\mathcal{F}, \mathcal{F}')|\mathcal{F}' \subset \mathcal{F}\}$$

Proof Consider the composition

$$(\mathcal{F}' \hookrightarrow \mathcal{O}^n \to \mathcal{O}^n/\mathcal{F}) \in \text{Hom}(\mathcal{F}', \mathcal{O}^n/\mathcal{F})$$

where the first map is the inclusion of $\mathcal{F}'$ in $\mathcal{O}^n$, and the second is projection onto the quotient. This morphism vanishes above $\infty \in \mathbb{P}^1$, so it naturally extends to a morphism $r(\mathcal{F}, \mathcal{F}') \in \text{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F})$. Thus we have obtained a section r of the bundle E, which is easily seen to vanish precisely when $\mathcal{F}' \subset \mathcal{F}$. Therefore, the zero locus of the section r is $\mathfrak{C}$. Since the rank of E is half the dimension of the base $\mathcal{M} \times \mathcal{M}$ (by Proposition 5.19), then the dimension of each irreducible component of the zero locus $\mathfrak{C}$ will be at least half the dimension of $\mathcal{M} \times \mathcal{M}$. Together with Proposition 4.15, this implies:

Proposition 6.10. *The distinct irreducible components of $\mathfrak{C}$ are precisely the sub-varieties $\mathfrak{C}_\kappa$, and all have dimension equal to half the dimension of $\mathcal{M} \times \mathcal{M}$.*

Now to prove that the section is regular, we must prove that r is transversal at the generic point of the zero locus. In other words, the image of the section r and the image of the zero section must be transversal in the total space $\mathcal{E}$ of E. Given a point $(\mathcal{F}, \mathcal{F}') \in \mathfrak{C}_\kappa$ with $\mathcal{F}' \subset \mathcal{F}$, we have the following relation of tangent spaces

$$\mathcal{T}_{(\mathcal{F}, \mathcal{F}', 0)}\mathcal{E} = \mathcal{T}^h \oplus \mathcal{T}^v$$

In the above decomposition, $\mathcal{T}^h$ (the horizontal tangent space) is just the pull-back of $\mathcal{T}_{(\mathcal{F},\mathcal{F}')}\mathcal{M} \times \mathcal{M}$, while $\mathcal{T}^v$ (the vertical tangent space) is isomorphic to the bundle E itself. The horizontal tangent space is spanned by vectors tangent to the image of the zero section. It is therefore enough to show that the vertical tangent space is spanned by vectors tangent to the image of r, for the generic point $(\mathcal{F},\mathcal{F}') \in \mathfrak{C}_\kappa$. Let $p_v : \mathcal{T}^h \oplus \mathcal{T}^v \to \mathcal{T}^v \cong E|_{(\mathcal{F},\mathcal{F}')}$ be the projection onto the second factor. Then what we must show is that the map $p_v \circ r_*$ maps $T_{(\mathcal{F},\mathcal{F}')}\mathcal{M} \times \mathcal{M}$ surjectively onto $\mathcal{T}^v$, and this will be equivalent to transversality at $(\mathcal{F},\mathcal{F}')$.

By Proposition 5.3, a tangent vector to $\mathcal{M}$ at $\mathcal{F}'$ is a homomorphism $\psi \in \operatorname{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F}')$. Such a vector is also tangent to $\mathcal{M} \times \mathcal{M}$ at $(\mathcal{F},\mathcal{F}')$; more precisely, it is tangent in the direction of the second factor of $\mathcal{M} \times \mathcal{M}$. The map $p_v \circ r_*$ maps ψ to the homomorphism

$$\pi \circ \psi = \mathcal{F}'(\infty) \to \mathcal{O}^n/\mathcal{F}' \to \mathcal{O}^n/\mathcal{F} \in \operatorname{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F})$$

where $\pi : \mathcal{O}^n/\mathcal{F}' \to \mathcal{O}^n/\mathcal{F}$ is the standard projection induced by the inclusion $\mathcal{F}' \hookrightarrow \mathcal{F}$. Then it is enough to show that the natural map

$$\operatorname{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F}') \xrightarrow{\psi \to \pi \circ \psi} \operatorname{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F})$$

is surjective for the generic point $(\mathcal{F},\mathcal{F}') \in \mathfrak{C}_\kappa$. Take an element $\phi \in \operatorname{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F})$, which corresponds to a commutative diagram of homomorphisms $(\phi_1, ..., \phi_{n-1})$ as in (5.2). Let $\pi_i : \mathcal{O}^n/\mathcal{F}'_i \to \mathcal{O}^n/\mathcal{F}_i$ be the standard projections. We have the exact sequence

$$\operatorname{Hom}(\mathcal{F}'_i(\infty), \mathcal{O}^n/\mathcal{F}'_i) \to \operatorname{Hom}(\mathcal{F}'_i(\infty), \mathcal{O}^n/\mathcal{F}_i) \to \operatorname{Ext}^1(\mathcal{F}'_i(\infty), \mathcal{F}_i/\mathcal{F}'_i) = 0$$

The Ext space on the right vanishes because $\mathcal{F}'_i(\infty)$ is a torsion free sheaf on $\mathbb{P}^1$, whereas $\mathcal{F}_i/\mathcal{F}'_i$ is a direct sum of skyscraper sheaves. Therefore, each homomorphism $\phi_i \in \operatorname{Hom}(\mathcal{F}'_i(\infty), \mathcal{O}^n/\mathcal{F}_i)$ can be extended to an element $\psi_i \in \operatorname{Hom}(\mathcal{F}'_i(\infty), \mathcal{O}^n/\mathcal{F}'_i)$ such that $\phi_i = \pi_i \circ \psi_i$. In this way, we can extend the homomorphism ϕ to a homomorphism $\psi \in \operatorname{Hom}(\mathcal{F}'(\infty), \mathcal{O}^n/\mathcal{F}')$ such that $\phi = \pi \circ \psi$. The only problem is that, a priori, $\psi = (\psi_1, ..., \psi_{n-1})$ does not make the diagram

$$
\begin{array}{ccc}
\cdots & & \cdots \\
\downarrow & & \downarrow \\
\mathcal{F}'_i(\infty) & \xrightarrow{\psi_i} & \mathcal{O}^n/\mathcal{F}'_i \\
\rho_i \downarrow & & \nu_i \downarrow \\
\mathcal{F}'_{i+1}(\infty) & \xrightarrow{\psi_{i+1}} & \mathcal{O}^n/\mathcal{F}'_{i+1} \\
\downarrow & & \downarrow \\
\cdots & & \cdots
\end{array}
$$

commute (in the above, ρ_i is the inclusion and ν_i is induced by the inclusion). In other words, we would like to have $\psi_{i+1} \circ \rho_i - \nu_i \circ \psi_i = 0$. However, all that we can say a priori is that

$$\pi_{i+1} \circ (\psi_{i+1} \circ \rho_i - \nu_i \circ \psi_i) = \phi_{i+1} \circ \rho_i - \nu_i \circ \phi_i = 0,$$

because the commutative diagram (5.2) is known to commute. Therefore, $\psi_{i+1} \circ \rho_i - \nu_i \circ \psi_i$ is a homomorphism between $\mathcal{F}'_i(\infty)$ and $\mathcal{F}_{i+1}/\mathcal{F}'_{i+1}$, and we want

to perturb the ψ_i's in such a way as to make this homomorphism 0. We will perform this perturbation by inductively adding to each ψ_{i+1} a homomorphism $d\psi_{i+1} : \mathcal{F}'_{i+1}(\infty) \to \mathcal{F}_{i+1}/\mathcal{F}'_{i+1}$ such that

$$(\psi_{i+1} + d\psi_{i+1}) \circ \rho_i - \nu_i \circ \psi_i = 0.$$

It is clear that this perturbation preserves the property $\pi \circ \psi = \phi$. To do this, it is enough to show that every homomorphism between $\mathcal{F}'_i(\infty)$ and $\mathcal{F}_{i+1}/\mathcal{F}'_{i+1}$ can be written as $d\psi_{i+1} \circ \rho_i$. In other words, it is enough to prove the surjectivity of the natural restriction map

$$\mathrm{Hom}(\mathcal{F}'_{i+1}(\infty), \mathcal{F}_{i+1}/\mathcal{F}'_{i+1}) \to \mathrm{Hom}(\mathcal{F}'_i(\infty), \mathcal{F}_{i+1}/\mathcal{F}'_{i+1})$$

But the cokernel of this map is contained in

$$\mathrm{Ext}^1(\mathcal{F}'_{i+1}/\mathcal{F}'_i, \mathcal{F}_{i+1}/\mathcal{F}'_{i+1})$$

and therefore it is enough to show that this Ext group is 0. Recall that $\mathcal{F}_{i+1}/\mathcal{F}'_{i+1}$ is a torsion sheaf, supported at finitely many points. Moreover, $\mathcal{F}'_{i+1}/\mathcal{F}'_i$ has finitely many "torsion points", i.e. points of $\mathbb{A}^1 = \mathbb{P}^1 \backslash \infty$ above which there is torsion. In order for the above Ext^1 group to vanish, it would be enough to have the support of $\mathcal{F}_{i+1}/\mathcal{F}'_{i+1}$ and the torsion points of $\mathcal{F}'_{i+1}/\mathcal{F}'_i$ be two disjoint sets (just apply Serre duality). Therefore, Lemma 6.9 reduces to the following:

Lemma 6.11. *For the generic point $(\mathcal{F}, \mathcal{F}') \in \mathfrak{C}_\kappa$, the support of the sheaf $\mathcal{F}_{i+1}/\mathcal{F}'_{i+1}$ is disjoint from the set of torsion points of $\mathcal{F}'_{i+1}/\mathcal{F}'_i$.*

Proof Note that the condition in the statement of the lemma is an open condition. Therefore, since $\mathfrak{C}_\kappa$ is irreducible, it is enough to find at least one point with the desired property. We will prove this lemma for the case when κ is an indecomposable quiver representation, since the case when κ is a sum of such representations is completely analogous.

Suppose that κ is the indecomposable quiver representation $j \to j + 1 \to ... \to k \to 0$, and choose any flag $\mathcal{F} \in p(\mathfrak{C}_\kappa)$. The sections of the flag components $\mathcal{F}_i \subset \mathcal{O}^n$ are (locally) vectors $(P_1, ..., P_n)$ of polynomials in x. For each i, the inclusion $\mathcal{F}_{i-1} \hookrightarrow \mathcal{F}_i$ will be an inclusion of vector bundles in the vicinity of the generic point $\zeta \in \mathbb{A}^1$. Near such a point ζ, the bundle $\mathcal{F}_{j-1}$ will be cut out in $\mathcal{F}_j$ by a linear equation $Q_1 P_1 + ... + Q_n P_n = 0$. We can choose ζ such that there exist $(P_1, ..., P_n) \in \mathcal{F}_j$ which verify $x - \zeta \nmid Q_1 P_1 + ... + Q_n P_n$. Then define $\mathcal{F}'_i = \mathcal{F}_i$ for $i < j$ or $i > k$, whereas for $j \le i \le k$ define

$$\mathcal{F}'_i = \{(P_1, ..., P_n) \in \mathcal{F}_i, \text{ such that } x - \zeta | Q_1 P_1 + ... + Q_n P_n\}$$

Then one sees that $\mathcal{F}'_{i-1} \subset \mathcal{F}'_i$ for all i. Indeed, the only non-trivial inclusion is the one for $i = j$, but this one follows from the fact that $Q_1 P_1 + ... + Q_n P_n = 0$ for $(P_1, ..., P_n) \in \mathcal{F}_{j-1}$. The maps

$$\mathcal{F}_j/\mathcal{F}'_j \to ... \to \mathcal{F}_k/\mathcal{F}'_k \to 0$$

induce the quiver representation $j \to j + 1 \to ... \to k \to 0$. To see this, note that for $j \le i \le k - 1$, we have that $\mathcal{F}_i \supset \mathcal{F}_j$ and $\mathcal{F}_j$ contains an element $(P_1, ..., P_n)$ such that $x - \zeta \nmid Q_1 P_1 + ... + Q_n P_n$. Therefore $(P_1, ..., P_n) \notin \mathcal{F}'_{i+1}$, and thus $\mathcal{F}_i$ is not a subsheaf of $\mathcal{F}'_{i+1}$. Thus we have that

$$(\mathcal{F}, \mathcal{F}') \in \mathfrak{C}^\circ_\kappa \subset \mathfrak{C}_\kappa$$

Now to show that $(\mathcal{F}, \mathcal{F}')$ constructed in this way satisfies the conclusion of the lemma. Obviously, the sheaves $\mathcal{F}_{i+1}/\mathcal{F}'_{i+1}$ have non-empty support only for $j-1 \leq i \leq k-1$, and in that case the support consists only of ζ. On the other hand, we will show that for $j-1 \leq i \leq k-1$ the sheaf $\mathcal{F}'_{i+1}/\mathcal{F}'_i$ does not have ζ as a torsion point. Indeed, take $(P_1, ..., P_n) \in \mathcal{F}'_{i+1}$ such that $((x-\zeta)P_1, ..., (x-\zeta)P_n) \in \mathcal{F}'_i$. Since ζ is not a torsion point of $\mathcal{F}_{i+1}/\mathcal{F}_i$ (by assumption on ζ), then $(P_1, ..., P_n) \in \mathcal{F}_i$. However, since $(P_1, ..., P_n) \in \mathcal{F}'_{i+1}$ and $i+1 \leq k$, this means that $x - \zeta | Q_1 P_1 + ... + Q_n P_n$. Therefore, $(P_1, ..., P_n) \in \mathcal{F}'_i$ and thus the sheaf $\mathcal{F}'_{i+1}/\mathcal{F}'_i$ has no torsion at ζ.   □

This completes the proof of Lemma 6.9.

□

Now let us return to the proof of Proposition 6.7. Proposition 3.9 and Lemma 6.9 imply that

$$c(E, 0) = e(E) = \sum_{\kappa \in \text{Rep}(A_{n-1})} [\mathfrak{C}_\kappa]$$

Therefore, in the notation of Section 4.5, we have

$$A(0) = \sum_{\kappa \in \text{Rep}(A_{n-1})} p_*([\mathfrak{C}_\kappa] \cdot q^*) = \sum_{\kappa \in \text{Rep}(A_{n-1})} e_\kappa = \sum_{b_1, ..., b_\nu \geq 0} e_{b_1 \theta_1 + ... + b_\nu \theta_\nu}$$

where $\theta_\nu > ... > \theta_1$ denote the positive roots. Applying Theorem 4.16, we know that the e_κ satisfy the same multiplication rules as the corresponding elements of $U(A_{n-1})^{op}$. This and (2.12) imply that

$$A(0) = \sum_{b_1, ..., b_\nu \geq 0} \frac{e_{\theta_\nu}^{b_\nu}}{b_\nu!} \cdot ... \cdot \frac{e_{\theta_1}^{b_1}}{b_1!} = \prod_{\theta \in R^+} \exp(e_\theta).$$

□

6.4. When $m \in \mathbb{N}$, there is a direct way to factor $A(m)$ in terms of $g := A(0)$. For that, let $s : \mathcal{M} \to \mathcal{M}$ be the map

$$s(\mathcal{F}) = \mathcal{F}(-m)$$

In the above, $\mathcal{F}(-m)$ denotes the flag obtained from $\mathcal{F}$ by twisting each of its components by $\mathcal{O}(-m) = \mathcal{O}(-m \cdot 0_{\mathbb{P}^1})$. Then we have

Proposition 6.12. *For generic $\frac{a}{x}$,*

$$A(m) = y \cdot g \cdot s_*$$

for some non-zero constant $y \in H^*_{T \times \mathbb{C}^*}(pt)$.

Remark 6.13. The word generic is mentioned because the constant y has poles as a rational function of $\frac{a}{x}$.

Proof The easiest way to prove this proposition is to use equivariant localization. By Theorem 3.5, it is enough to prove that the desired equality holds for basis vectors $[d] \in H$, where $d = (d^i_j)$. Thus, we must show that

$$A(m)[d] = y \cdot A(0) \cdot s_*[d]$$

Obviously, $s_*[d] = [s(d)] = [d+m]$, where $d+m := (d_j^i + m)$. By using (6.2), the above becomes equivalent to

$$A(m)[d] = \sum_{d'}[d'] \cdot \frac{\prod\limits_{w \in E_{d',d}}(w+mx)}{\prod\limits_{w \in \mathcal{T}_{d'}\mathcal{M}} w} = y \cdot \sum_{d'}[d'] \cdot \frac{\prod\limits_{w \in E_{d',d+m}} w}{\prod\limits_{w \in \mathcal{T}_{d'}\mathcal{M}} w} = y \cdot A(0) \cdot s_*[d]$$

Therefore, we must show that for any fixed points d, d' we have

$$\prod_{w \in E_{d',d}}(w+mx) = y \cdot \prod_{w \in E_{d',d+m}} w$$

for some non-zero constant $y \in H_{T \times \mathbb{C}^*}^*(pt)$ which is independent of d, d'. Since in any representation of $T \times \mathbb{C}^*$, the character is the sum of the exponentials of the weights, the above relation is equivalent to

$$\text{char}(E_{d',d}) \cdot e^{mx} = \text{constant} + \text{char}(E_{d',d+m})$$

where the constant does not depend on d, d'. This immediately follows from Proposition 5.22.

$\square$

6.5. The map s_* has a number of nice commutation properties with the generators of the $\mathfrak{sl}_n-$action on H.

Proposition 6.14. *The map s_* commutes with $f_1, ..., f_{n-1}, e_1, ..., e_{n-2}$. Moreover, its commutator with h_{n-1} is given by*

$$[s_*, h_{n-1}] = mn \cdot s_* \tag{6.15}$$

Proof We will prove that s_* commutes with e_i, for $1 \le i \le n-2$. The proof of commutativity with the f_i's will be completely analogous. Recall that $e_i = \tilde{p}_*\tilde{q}^*$, where $\tilde{p}, \tilde{q}$ are the standard projections from $\mathfrak{C}_i \subset \mathcal{M} \times \mathcal{M}$ onto the two factors. Consider the correspondence

$$\mathfrak{C}_i' = \{(\mathcal{F}, \mathcal{F}') \in \mathcal{M} \times \mathcal{M} | \mathcal{F}'(-m) \subset_i \mathcal{F}\}$$

and let $\widehat{p}, \widehat{q} : \mathfrak{C}_i' \to \mathcal{M}$ be the standard projections. Moreover, let $s' : \mathfrak{C}_i' \to \mathfrak{C}_i$ be the map given by $s'(\mathcal{F}, \mathcal{F}') = (\mathcal{F}, \mathcal{F}'(-m))$.

From the definitions, it is easily seen that

$$
\begin{array}{ccccc}
\mathcal{M} & \xleftarrow{\;\widehat{p}\;} & \mathfrak{C}_i' & \xrightarrow{\;\widehat{q}'\;} & \mathcal{M} \\
{\scriptstyle \text{Id}}\downarrow & & {\scriptstyle s'}\downarrow & & {\scriptstyle s}\downarrow \\
\mathcal{M} & \xleftarrow{\;\tilde{p}\;} & \mathfrak{C}_i & \xrightarrow{\;\tilde{q}\;} & \mathcal{M}
\end{array}
$$

commutes, and moreover the square on the right is a fiber square. Therefore we have the base-change formula $\tilde{q}^* s_* = s'_* \widehat{q}^*$, and thus

$$e_i \cdot s_* = \tilde{p}_* \tilde{q}^* s_* = \tilde{p}_* s'_* \widehat{q}^* = \widehat{p}_* \widehat{q}^* \tag{6.16}$$

Now let $s'' : \mathfrak{C}_i \to \mathfrak{C}'_i$ be the map given by $s''(\mathcal{F}, \mathcal{F}') = (\mathcal{F}(-m), \mathcal{F}')$. It gives us the commutative diagram

$$
\begin{array}{ccccc}
\mathcal{M} & \xleftarrow{\tilde{p}} & \mathfrak{C}_i & \xrightarrow{\tilde{q}} & \mathcal{M} \\
{\scriptstyle s}\downarrow & & {\scriptstyle s''}\downarrow & & {\scriptstyle \mathrm{Id}}\downarrow \\
\mathcal{M} & \xleftarrow{\widehat{p}} & \mathfrak{C}'_i & \xrightarrow{\widehat{q}} & \mathcal{M}
\end{array}
$$

There is no fiber square here, but note that s'' is an isomorphism, because the inverse map $(\mathcal{F}, \mathcal{F}') \to (\mathcal{F}(m), \mathcal{F}')$ is well-defined. Note that this only holds for $1 \le i \le n-2$, since otherwise we cannot guarantee the fact that $\mathcal{F}(m) \subset \mathcal{O}^n$.

Since s'' is an isomorphism, therefore $s''_* s''^* = \mathrm{Id}$. The above commutative diagram then gives us

$$
s_* \cdot e_i = s_* \tilde{p}_* \tilde{q}^* = \widehat{p}_* s''_* \tilde{q}^* = \widehat{p}_* s''_* s''^* \widehat{q}^* = \widehat{p}_* \widehat{q}^* \tag{6.17}
$$

Thus (6.16) and (6.17) imply that $e_i \cdot s_* = \widehat{p}_* \widehat{q}^* = s_* \cdot e_i$.

To prove (6.15), we will check it on any basis vector $[d]$, where $d = (d^i_j)$ is a vector of non-negative integers. Since s takes the fixed point d to the fixed point $d + m = (d^i_j + m)$, we will have $s_*[d] = [d + m]$. The explicit formula for h_{n-1} in (4.11) gives us

$$
[s_*, h_{n-1}]([d]) = \left(\frac{a_{n-1}}{x} - \frac{a_n}{x} - 1 + d_{n-2} - 2d_{n-1} \right) \cdot [d + m] -
$$

$$
- \left(\frac{a_{n-1}}{x} - \frac{a_n}{x} - 1 + d_{n-2} - 2d_{n-1} - mn \right) \cdot [d + m] = mn \cdot s_*[d]
$$

$\square$

The properties of s_* listed in the previous proposition are very important, because they completely determine the map s_* (up to a constant).

Proposition 6.18. *All maps* $\Psi : H \to H$ *which commute with the action of* $f_1, ..., f_{n-1}, e_1, ..., e_{n-2}$ *and satisfy*

$$
[\Psi, h_{n-1}] = mn \cdot \Psi
$$

are constant multiples of s_*.

Proof Let us first compute $\Psi([0])$, where $0 = (0, ..., 0)$. We will write $\Psi([0])$ in the basis of cohomology classes $[d]$:

$$
\Psi([0]) = \sum_d \alpha_d \cdot [d] \tag{6.19}
$$

The operators $h_1, ..., h_{n-1}$ are diagonal in the basis $[d]$. Recall that Ψ commutes with $h_1, ..., h_{n-2}$ and $[\Psi, h_{n-1}] = mn \cdot \Psi$. Therefore, (4.11) implies that the d's that appear with non-zero coefficient in the above sum have degrees $d_1 = m$, $d_2 = 2m$, ..., $d_{n-1} = (n-1)m$.

Now suppose that there is a $\tilde{d}$ such that $\alpha_{\tilde{d}} \neq 0$ and $\tilde{d}^i_{j+1} < \tilde{d}^i_j$ for some $j \le n-2$. If there are several such $\tilde{d}$'s, choose the one with the smallest j, and if there are

several such $\tilde{d}$'s for the same j, then choose the one such that the number $\tilde{d}_j^i$ is minimal among all i. Since Ψ commutes with e_j, it follows that

$$e_j \left(\sum_d \alpha_d \cdot [d] \right) = 0$$

The coefficient of $[\tilde{d} - \delta_j^i]$ in the sum from the left hand side can only come from $e_j[\tilde{d}]$, by minimality of $\tilde{d}_j^i$. Since $\tilde{d}_{j+1}^i < \tilde{d}_j^i$, this coefficient is non-zero (see for example the computation of matrix coefficients of e_j in [6]). Therefore, we obtain a contradiction with the assumption that $\alpha_{\tilde{d}} \neq 0$.

This means that in (6.19), we can have $\alpha_d \neq 0$ only if $d_j^i = d_{j+1}^i$ for all $1 \leq i \leq j \leq n-2$. Since $d_1 = m$, $d_2 = 2m$, ..., $d_{n-1} = m(n-1)$, the only d that can appear with non-zero coefficient in (6.19) is the one with $d_j^i = m, \forall i, j$. Therefore,

$$\Psi([0]) = \alpha \cdot [m] = \alpha \cdot s_*[0]$$

for some $\alpha \in \mathbb{C}$. But recall that H is isomorphic to the Verma module, and thus the f_i's generate H from the vector $[0]$. Since both Ψ and s_* commute with the f_i's, we will therefore have the identity of operators $\Psi = \alpha \cdot s_*$. $\square$

6.6. Let S^{mn} denote the mn−th symmetric power of the dual to the standard representation of $\mathfrak{sl}_n$ in $\mathbb{C}^n$. As a vector space, S^{mn} consists of homogeneous polynomials of degree mn in variables $y_1, ..., y_n$. The $\mathfrak{sl}_n$−action on S^{mn} is given by

$$e_i = -y_{i+1} \frac{\partial}{\partial y_i}, \quad f_i = -y_i \frac{\partial}{\partial y_{i+1}}$$

$$h_i = y_{i+1} \frac{\partial}{\partial y_{i+1}} - y_i \frac{\partial}{\partial y_i}$$

There is a map $\tilde{ev} : S^{mn} \to \mathbb{C}$ given by evaluating polynomials at $y_1 = ... = y_{n-1} = 0, y_n = 1$. Then $\tilde{ev}$ annihilates all monomials except for y_n^{mn}. If we give $\mathbb{C}$ the trivial $\mathfrak{sl}_n$−action, then one easily computes the fact that $\tilde{ev}$ commutes with the actions of $f_1, ..., f_{n-1}, e_1, ..., e_{n-2}$. Moreover, we have that $[\tilde{ev}, h_{n-1}] = mn \cdot \tilde{ev}$. Therefore the map

$$\text{Id} \otimes \tilde{ev} : H \otimes S^{mn} \to H \otimes \mathbb{C} = H \tag{6.20}$$

commutes with the actions of $f_1, ..., f_{n-1}, e_1, ..., e_{n-2}$ and satisfies $[\text{Id} \otimes \tilde{ev}, h_{n-1}] = mn \cdot \text{Id} \otimes \tilde{ev}$. As in [14], up to a constant multiple there is a unique $\mathfrak{sl}_n$−intertwiner $\Phi_m : H \to H \otimes S^{mn}$. By the above, the composition

$$(\text{Id} \otimes \tilde{ev}) \cdot \Phi_m : H \to H \tag{6.21}$$

commutes with the actions of $f_1, ..., f_{n-1}, e_1, ..., e_{n-2}$ and satisfies $[(\text{Id} \otimes \tilde{ev}) \cdot \Phi_m, h_{n-1}] = mn \cdot (\text{Id} \otimes \tilde{ev}) \cdot \Phi_m$. Therefore, Proposition 6.18 immediately implies the following:

Proposition 6.22. *We have*

$$s_* = \alpha \cdot (\text{Id} \otimes \tilde{ev}) \cdot \Phi_m$$

for some constant $\alpha \in \mathbb{C}$.

6.7. From Proposition 6.12, we have that $A(m) = y \cdot g \cdot s_*$. Proposition 6.22 implies that

$$A(m) = (y\alpha) \cdot g \cdot (\text{Id} \otimes \tilde{ev}) \cdot \Phi_m \tag{6.23}$$

Since $g \in SL_n$, it acts in a group-like fashion on tensor products, and thus

$$g \cdot (\text{Id} \otimes \tilde{ev}) \cdot g^{-1} = \text{Id} \otimes (g \cdot \tilde{ev} \cdot g^{-1}) \tag{6.24}$$

A trivial check reveals that $g \cdot \tilde{ev} \cdot g^{-1} = ev$, where $ev : S^{mn} \to \mathbb{C}$ is the linear map that evaluates polynomials at $y_1 = ... = y_n = 1$. Therefore we have that $g \cdot (\text{Id} \otimes \tilde{ev}) = (\text{Id} \otimes ev) \cdot g$. Putting this and (6.23) together implies that

$$A(m) = (y\alpha) \cdot (\text{Id} \otimes ev) \cdot g \cdot \Phi_m \tag{6.25}$$

Since Φ_m denotes any intertwiner, we can absorb the constant $y\alpha$ into it, and then make it commute with g in (6.25). Therefore, we obtain the following:

Theorem 6.26. *For generic $\frac{a}{x}$, the operator $A(m)$ has the factorization*

$$A(m) : H \xrightarrow{g} H \xrightarrow{\Phi_m} H \otimes S^{mn} \xrightarrow{\text{Id} \otimes ev} H$$

where $g = A(0)$ and Φ_m is an $\mathfrak{sl}_n-$intertwiner.

7. The generating function $Z(m)$ and the Calogero-Sutherland hamiltonian

7.1. As in Theorem 4.9, we have $H \cong M(\frac{a}{x} - \rho)$. Therefore, Theorem 6.26 gives us

$$A(m) = (\text{Id} \otimes ev) \cdot \Phi_m \cdot g$$

where $\Phi_m : M(\frac{a}{x} - \rho) \to M(\frac{a}{x} - \rho) \otimes S^{mn}$ is an interwtiner. By Theorem 2 in [14], the character of Φ_m is given by

$$\chi_{\Phi_m} = c \cdot Y_{\frac{a}{x},m} \cdot \delta^{-1} \cdot (y_1...y_n)^m \tag{7.1}$$

In the above, $Y_{\frac{a}{x},m}$ denotes the eigenfunction of the Calogero-Sutherland hamiltonian as in (2.10). Moreover, δ is the Weyl denominator (2.7) and $c \in H^*_{T \times \mathbb{C}^*}(pt)$ is a constant. To compute the character of $A(m)$, we will use the fact that it is determined by its values on elements e^h of the maximal torus:

$$\chi_{A(m)}(e^h) = \text{Tr}((\text{Id} \otimes ev) \cdot \Phi_m \cdot ge^h) = ev(\text{Tr}(\Phi_m \cdot ge^h)) = ev(\chi_{\Phi_m}(ge^h)) \tag{7.2}$$

If we denote $e^h = \text{diag}(e^{h_1}, ..., e^{h_n})$, then

$$ge^h = \begin{pmatrix} e^{h_1} & e^{h_2} & e^{h_3} & ... & e^{h_n} \\ 0 & e^{h_2} & e^{h_3} & ... & e^{h_n} \\ 0 & 0 & e^{h_3} & ... & e^{h_n} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & e^{h_n} \end{pmatrix}$$

One easily diagonalizes the above matrix and notices that $ge^h = x^{-1}e^h x$, where the matrix x is unipotent upper triangular, whose entry on row i and column j is

$$x_{ij} = \frac{1}{e^{h_i - h_j} - 1} \cdot \prod_{k=i+1}^{j-1} \frac{1}{1 - e^{h_k - h_i}}$$

The equivariance of χ_{Φ_m} (relation (2.5)) and (7.2) yield

$$\chi_{A(m)}(e^h) = ev(\chi_{\Phi_m}(x^{-1}e^h x)) = ev(x^{-1} \cdot \chi_{\Phi_m}(e^h))$$

By (7.1), the above becomes

$$\chi_{A(m)}(e^h) = c \cdot Y_{\frac{a}{x},m}(e^h) \cdot \delta^{-1}(e^h) \cdot ev(x^{-1} \cdot (y_1...y_n)^m) \qquad (7.3)$$

Since the representation S^{mn} is nothing other than the $mn-$th symmetric power of the dual of $\mathbb{C}^n$, to compute $ev(x^{-1} \cdot (y_1...y_n)^m)$ we must apply the matrix x to the vector $(1, 1, ..., 1)$, multiply the entries and raise the result to the $m-$th power. Thus we find that

$$ev(x^{-1} \cdot (y_1...y_n)^m) = \left(\prod_{n \geq j > i \geq 1} \frac{1}{1 - e^{h_j - h_i}} \right)^m =$$

$$= \left(\prod_{\alpha \in R^+} \frac{1}{1 - e^{\langle h, -\alpha \rangle}} \right)^m = \left(e^{\langle h, \rho \rangle} \prod_{\alpha \in R^+} \frac{1}{e^{\langle h, \alpha/2 \rangle} - e^{\langle h, -\alpha/2 \rangle}} \right)^m = \frac{e^{\langle h, m\rho \rangle}}{\delta^m(e^h)}$$

Plugging thus into (7.3) gives us the equality of formal power series

$$\chi_{A(m)} = c \cdot Y_{\frac{a}{x},m} \cdot \delta^{-1} \cdot \frac{e^{m\rho}}{\delta^m} = c \cdot e^{m\rho} \cdot Y_{\frac{a}{x},m} \cdot \delta^{-m-1} \qquad (7.4)$$

From the above relation, the constant c is precisely the coefficient of $e^{\frac{a}{x} - \rho}$ in the power series $\chi_{A(m)}$. By (6.4), this equals

$$c = \prod_{w \in \mathcal{T}_0 \mathcal{M}} \frac{w + mx}{w}$$

where 0 is the torus fixed flag with all degrees 0. But the point 0 is isolated in $\mathcal{M}$ (since the component in which it lies has dimension 0), and therefore $c = 1$. Thus (7.4) and Proposition 6.6 imply the following:

Theorem 7.5. *The generating function $Z(m)$ satisfies*

$$Z(m) = e^{-\frac{a}{x} + (m+1)\rho} \cdot Y_{\frac{a}{x},m} \cdot \delta^{-m-1} = Y_{\frac{a}{x},m} \cdot e^{-\frac{a}{x}} \cdot \left(\prod_{\alpha \in R^+} \frac{1}{1 - e^{-\alpha}} \right)^{m+1} \qquad (7.6)$$

Here $Y_{\frac{a}{x},m}$ is the eigenfunction of the Calogero-Sutherland hamiltonian

$$L(m) \cdot Y_{\frac{a}{x},m} = \frac{(a, a)}{x^2} \cdot Y_{\frac{a}{x},m}$$

with highest term $e^{\frac{a}{x}}$.

Remark 7.7. Technically, the above proof of Theorem 7.5 only holds for natural numbers m and for generic $\frac{a}{x}$. However, the coefficients of $Z(m)$ and $Y_{\frac{a}{x},m}$ are rational functions of both m and $\frac{a}{x}$. Therefore, since the equality (7.6) holds for infinitely many m and for generic $\frac{a}{x}$, it holds identically. Similarly, once the factorization Theorem 6.26 is proven for natural numbers m, it holds identically in m (one must just be a bit careful with the definition of S^{mn} for non-natural m, which will be an infinite dimensional space).

7.2. In the rest of the paper, we will take the limit $m \to \infty$ of Theorem 7.5 and obtain the finite-dimensional statement of Corollary 3.7 of [4]. Let us henceforth write $m = e^P x^{-1}$, where $P \in \mathbb{C}$. To simplify notations, we will write $Y_P = Y_{\frac{a}{x}, m}$. Consider the element $\rho^\vee \in \mathfrak{h}$ such that $\langle \rho^\vee, \alpha_i \rangle = 1$ for any simple root α_i. In this section we will use two sets of coordinates on $\mathfrak{h}$: the *old coordinate h*, and the *new coordinate* given by $h' = h + 2P\rho^\vee$. Under this coordinate transformation, the symbol e^λ transforms to $e^\lambda \cdot e^{2P\langle \rho^\vee, \lambda \rangle}$. Recall from Section 2.4 that in the old coordinate h we have

$$Y_P = \sum_{\lambda \in \frac{a}{x} + Q^-} e^\lambda c_{\lambda, P}$$

where we write $c_{\lambda, P}$ to emphasize the dependence on P. As in Section 2.4, the coefficients $c_{\lambda, P}$ are given by $c_{\frac{a}{x}, P} = 1$ and

$$c_{\lambda, P} \cdot \left((\lambda, \lambda) - \frac{(a, a)}{x^2} \right) = \frac{2e^P}{x} \left(\frac{e^P}{x} + 1 \right) \sum_{\alpha \in R^+} \sum_{j \geq 1} j \cdot c_{\lambda + j\alpha, P} \tag{7.8}$$

We assume generic $\frac{a}{x}$. For $\lambda = \frac{a}{x} - (d_1\alpha_1 + ... + d_{n-1}\alpha_{n-1})$ we have

$$d_1 + ... + d_{n-1} = \langle \rho^\vee, -\lambda \rangle + \langle \rho^\vee, \frac{a}{x} \rangle.$$

From (7.8), one can prove by induction on λ that $c_{\lambda, P} \sim e^{2P(d_1 + ... + d_{n-1})}$, i.e.

$$\lim_{P \to \infty} \frac{c_{\lambda, P}}{e^{2\langle \rho^\vee, -\lambda \rangle} \cdot e^{2\langle \rho^\vee, \frac{a}{x} \rangle}} = \lim_{P \to \infty} \frac{c_{\lambda, P}}{e^{2P(d_1 + ... + d_{n-1})}} =: c_\lambda \in \mathbb{C} \tag{7.9}$$

Writing out the power series Y_P in the new coordinate h' gives us

$$Y_P = e^{2P\langle \rho^\vee, \frac{a}{x} \rangle} \cdot \sum_{\frac{a}{x} + \lambda \in Q^-} e^\lambda \tilde{c}_{\lambda, P}$$

where $\tilde{c}_{\lambda, P} = c_{\lambda, P} \cdot e^{2P\langle \rho^\vee, \lambda \rangle} \cdot e^{-2P\langle \rho^\vee, \frac{a}{x} \rangle}$. Therefore relation (7.9) implies that

$$\lim_{P \to \infty} \tilde{c}_{\lambda, P} = c_\lambda$$

This implies that we have the following termwise convergence of power series, in the new coordinate:

$$\lim_{P \to \infty} e^{-2P\langle \rho^\vee, \frac{a}{x} \rangle} \cdot Y_P = \sum_{\lambda \in \frac{a}{x} + Q^-} e^\lambda c_\lambda =: Y \tag{7.10}$$

Since in the new coordinate h' the functions $e^{-2P\langle \rho^\vee, \frac{a}{x} \rangle} \cdot Y_P$ have highest term $e^{\frac{a}{x}}$, then Y will be a power series with highest term $e^{\frac{a}{x}}$. As described in [13], in the new coordinate h' we have

$$L(e^P) = \Delta_\mathfrak{h} - \frac{2e^P}{x} \left(\frac{e^P}{x} + 1 \right) \sum_{\alpha \in R^+} \frac{1}{e^\alpha \cdot e^{2P\langle \rho^\vee, \alpha \rangle} (1 - e^{-\alpha} \cdot e^{-2P\langle \rho^\vee, \alpha \rangle})^2}$$

For $\alpha \in R^+$ we have $\langle \rho^\vee, \alpha \rangle \geq 1$, with equality if and only if α is a simple root. Thus as $P \to \infty$, the summands in the above expression only survive if α is a simple root, otherwise they go to zero. Therefore, in the new coordinate,

$$L = \lim_{P \to \infty} L(e^P) = \Delta_\mathfrak{h} - \frac{2}{x^2} \sum_{\alpha \text{ simple}} e^{-\alpha}$$

This operator L is precisely the *quantum Toda Hamiltonian* (see [4]). Because the eigenfunctions $e^{-2P\langle \rho^\vee, \frac{a}{x} \rangle} \cdot Y_P$ converge as $P \to \infty$, their limit Y will be the

unique eigenfunction of L with eigenvalue $\frac{(a,a)}{x^2}$ and highest term $e^{\frac{a}{x}}$.

7.3. Keep the notation $m = e^P x^{-1}$ from the previous section. Recall from (6.5) that (in the old coordinate h):

$$Z(e^P x^{-1}) = \sum_{\gamma \in Q^-} e^\gamma \sum_{d \in \mathcal{M}_\gamma^{T \times \mathbb{C}^*}} \prod_{w \in \mathcal{T}_d \mathcal{M}} \frac{w + e^P}{w}$$

In the new coordinate h', the above becomes:

$$Z(e^P x^{-1}) = \sum_{\gamma \in Q^-} e^\gamma \cdot e^{2P \langle \rho^\vee, \gamma \rangle} \sum_{d \in \mathcal{M}_\gamma^{T \times \mathbb{C}^*}} \prod_{w \in \mathcal{T}_d \mathcal{M}} \frac{w + e^P}{w}$$

But $2\langle \rho^\vee, -\gamma \rangle$ is precisely the dimension of $\mathcal{M}_\gamma$, and therefore

$$Z(e^P x^{-1}) = \sum_{\gamma \in Q^-} e^\gamma \sum_{d \in \mathcal{M}_\gamma^{T \times \mathbb{C}^*}} \prod_{w \in \mathcal{T}_d \mathcal{M}} \frac{w + e^P}{w e^P}$$

Letting $P \to \infty$ we notice that the above series converges termwise

$$\lim_{P \to \infty} Z(e^P x^{-1}) = \sum_{\gamma \in Q^-} e^\gamma \sum_{d \in \mathcal{M}_\gamma^{T \times \mathbb{C}^*}} \prod_{w \in \mathcal{T}_d \mathcal{M}} \frac{1}{w} = \sum_{\gamma \in Q^-} e^\gamma \int_{\mathcal{M}_\gamma} 1 = Z$$

where Z is given by (1.3). In the above relation, the middle equality is proven by the equivariant integration result in Corollary 3.8. In the old coordinate h, Theorem 7.5 tells us that

$$Z(e^P x^{-1}) = Y_P \cdot e^{-\frac{a}{x}} \cdot \left(\prod_{\alpha \in R^+} \frac{1}{1 - e^{-\alpha}} \right)^{e^P x^{-1} + 1}$$

In the new coordinate, this equality becomes

$$Z(e^P x^{-1}) = Y_P \cdot e^{-\frac{a}{x}} \cdot e^{-2P \langle \rho^\vee, \frac{a}{x} \rangle} \cdot \left(\prod_{\alpha \in R^+} \frac{1}{1 - e^{-\alpha} \cdot e^{-2P \langle \rho^\vee, \alpha \rangle}} \right)^{e^P x^{-1} + 1}$$

Since $\langle \rho^\vee, \alpha \rangle \geq 1$ for every $\alpha \in R^+$, as $P \to \infty$ the last factor in the above product converges to 1. Taking limits as $P \to \infty$ in the above relation therefore gives us

$$Z = \lim_{P \to \infty} Z(e^P x^{-1}) = e^{-\frac{a}{x}} \cdot \lim_{P \to \infty} e^{-2P \langle \rho^\vee, \frac{a}{x} \rangle} \cdot Y_P$$

By equation (7.10), the right hand side in the above is nothing but $e^{-\frac{a}{x}} \cdot Y$. Therefore, we obtain the finite-dimensional statement of Corollary 3.7 in [4]:

Corollary 7.11. *The generating function Z of (1.3) equals $Y \cdot e^{-\frac{a}{x}}$, where Y is the eigenfunction of the quantum Toda hamiltonian with eigenvalue $\frac{(a,a)}{x^2}$ and highest term $e^{\frac{a}{x}}$.*

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